ON SIMULTANEOUS DIAGONAL INEQUALITIES, II

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1. Introduction

In this paper we continue the investigation begun in [11]. Let $\lambda_1, \dots, \lambda_s$ and $\mu_1, \dots, \mu_s$ be real numbers, and define the forms

$$F(\mathbf{x}) = \lambda_1 x_1^3 + \cdots + \lambda_s x_s^3,$$
$$G(\mathbf{x}) = \mu_1 x_1^2 + \cdots + \mu_s x_s^2.$$

Further, let τ be a positive real number. Our goal is to determine conditions under which the system of inequalities

$$|F(\mathbf{x})| < \tau, \qquad |G(\mathbf{x})| < \tau \tag{1.1}$$

has a non-trivial integral solution. As has frequently been the case in work on systems of diophantine inequalities (see for example Brüdern and Cook [6] and Cook [7]), we were forced in [11] to impose a condition requiring certain coefficient ratios to be algebraic. A recent paper of Bentkus and Götze [4] introduced a method for avoiding such a restriction in the study of positive-definite quadratic forms, and these ideas are in fact flexible enough to be applied to other problems. In particular, Freeman [10] was able to adapt the method to obtain an asymptotic lower bound for the number of solutions of a single diophantine inequality, thus finally providing the expected strengthening of a classical theorem of Davenport and Heilbronn [9]. The purpose of the present note is to apply these new ideas to the system of inequalities (1.1).

Write $|\mathbf{x}| = \max(|x_1|, \dots, |x_s|)$, and let $N(P)$ denote the number of integral solutions of the system (1.1) satisfying $|\mathbf{x}| \le P$. We establish the following result.

Theorem 1. *Suppose that $s \ge 13$, and let $\lambda_1, \dots, \lambda_s$ and $\mu_1, \dots, \mu_s$ be real numbers such that for some i and j both of the ratios λ_i/λ_j and μ_i/μ_j are irrational. Then one has*

$$N(P) \gg P^{s-5},$$

provided that

(i) *the form $F(\mathbf{x})$ has at least $s-4$ variables explicit,*
(ii) *the form $G(\mathbf{x})$ has at least $s-5$ variables explicit, and*
(iii) *the system of equations $F(\mathbf{x}) = G(\mathbf{x}) = 0$ has a non-singular real solution.*

In [11], we actually provided a more quantitative conclusion, in which the parameter τ in (1.1) was replaced by an explicit function of $|\mathbf{x}|$. Specifically, it was shown that when the coefficients of F and G are algebraic then under the conditions of Theorem 1 the system

$$|F(\mathbf{x})| < |\mathbf{x}|^{-\sigma_1}, \qquad |G(\mathbf{x})| < |\mathbf{x}|^{-\sigma_2} \tag{1.2}$$

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has at least on the order of $P^{s-5-\sigma_1-\sigma_2}$ integer solutions in the box $|\mathbf{x}| \leq P$, provided that $\sigma_1 + \sigma_2 < 1/12$. Results of this nature do not appear to be obtainable in the current state of knowledge without the restriction to algebraic coefficients. The reason is that the permissible size of $\sigma_1 + \sigma_2$ is determined by the amount of "excess" savings one generates on the minor arcs, but the Bentkus-Götze-Freeman method produces minor arc estimates of the form $o(P^{s-5})$ without giving any further indication of the actual order of magnitude.

Our proof of Theorem 1 is modeled on the paper of Freeman [10]. As usual, the main challenge is to handle the minor arcs. We first seek to demonstrate that large Weyl sums yield good rational approximations to the coefficients of F and G. This is essentially a theorem of R. Baker [1] (see also [2] and [3]), but we require a slightly sharper version, which we establish in §2. Then in §3 we are able to apply a two-dimensional version of Freeman's argument, combined with the methods of [11], to complete the proof. It should be noted that much of the analysis underlying the results quoted from [11] dates to the work of Wooley [13], [14] on simultaneous additive equations.

The author is grateful to Eric Freeman for alerting him to the work of Bentkus and Götze [4], for supplying a preprint of his own paper [10], and for helpful discussions of these important new ideas.

2. Diophantine Approximation Via Large Weyl Sums

As usual, we adopt the notation $e(z) = e^{2\pi iz}$. From a result of Baker [1], we know that whenever the exponential sum

$$F(\boldsymbol{\alpha}) = \sum_{1 \leq x \leq P} e(\alpha_3 x^3 + \alpha_2 x^2)$$

is large, one obtains good simultaneous rational approximations to the coefficients α_2 and α_3. Unfortunately, the bound for the denominator of these approximations fails by a factor of P^ε to allow us to initiate a minor arc analysis along the lines of Freeman [10]. Therefore we provide the following slight refinement, in the spirit of [10], Lemma 2.

Lemma 2.1. *Let ε be a positive real number, and suppose that P is sufficiently large in terms of ε. Suppose that $|F(\boldsymbol{\alpha})| \geq \gamma^{1/8} P$, where $P^{-1/64} \leq \gamma \leq 1$. Then there is a positive integer q, integers a_2 and a_3 satisfying $(q, a_2, a_3) = 1$, and absolute constants c_0, c_2, and c_3, such that*

$$q \leq c_0 \gamma^{-65}, \quad |q\alpha_2 - a_2| \leq c_2 \gamma^{-2} P^{-2+\varepsilon}, \quad \textit{and} \quad |q\alpha_3 - a_3| \leq c_3 \gamma^{-9} P^{-3}.$$

Proof. We follow Baker's argument fairly closely, deviating only as necessary to save the factor of P^ε in the bound for q. In view of the conclusions of the lemma, we may assume throughout that ε is sufficiently small. We have by a trivial extension of Freeman [10], Lemma 2, that there exists a positive integer r, an integer b, and absolute constants C_0 and C_3 satisfying

$$(b, r) = 1, \quad r < C_0 \gamma^{-64}, \quad \text{and} \quad |r\alpha_3 - b| < C_3 \gamma^{-8} P^{-3}. \tag{2.1}$$

Applying Weyl differencing, we find that

$$|F(\boldsymbol{\alpha})|^2 \leq \sum_{|h| < P} |S(h)|,$$

where

$$S(h) = S(h;\boldsymbol{\alpha}) = \sum_{1\le x\le P-h} e(3h\alpha_3 x^2 + (3h^2\alpha_3 + 2h\alpha_2)x). \tag{2.2}$$

Trivially, we have $|S(h)| \le P$ for all h, so

$$\sum_{|h|\le \frac{1}{5}\gamma^{1/4}P} |S(h)| \le \left(\tfrac{2}{5}\gamma^{1/4}P + 1\right) P \le \tfrac{1}{2}\gamma^{1/4}P^2$$

for P sufficiently large, and thus

$$\sum_{\frac{1}{5}\gamma^{1/4}P<|h|<P} |S(h)| \;\ge\; |F(\boldsymbol{\alpha})|^2 - \tfrac{1}{2}\gamma^{1/4}P^2 \;\ge\; \tfrac{1}{2}\gamma^{1/4}P^2. \tag{2.3}$$

Now the number of divisors of r is $O(r^{1/256})$, so on using (2.1) and (2.3) we find that

$$\tfrac{1}{2}\gamma^{1/4}P^2 \;\le\; \sum_{d|r} \sum_{\substack{\frac{1}{5}\gamma^{1/4}P<|h|<P \\ (h,r)=d}} |S(h)| \;\le\; c\gamma^{-1/4} \sum_{\substack{\frac{1}{5}\gamma^{1/4}P<|h|<P \\ (h,r)=D}} |S(h)|$$

for some D dividing r and some absolute constant c. It follows that

$$\sum_{\substack{\frac{1}{5}\gamma^{1/4}P<|h|<P \\ (h,r)=D}} |S(h)| \;\ge\; (2c)^{-1}\gamma^{1/2}P^2.$$

Moreover, on putting $C = (8c)^{-1}$, we see that the terms for which $|S(h)| \le C\gamma^{1/2}P$ contribute at most $2C\gamma^{1/2}P^2$ to this sum, so on writing

$$\mathcal{B} = \{h : \tfrac{1}{5}\gamma^{1/4}P < |h| < P,\ (h,r) = D,\ \text{and } |S(h)| > C\gamma^{1/2}P\},$$

we find that

$$\sum_{h\in\mathcal{B}} |S(h)| \;\ge\; 2C\gamma^{1/2}P^2,$$

and it follows that

$$\mathrm{card}(\mathcal{B}) \ge 2C\gamma^{1/2}P. \tag{2.4}$$

Choose any $h \in \mathcal{B}$, and put $b_3 = 3hb$. Then since $|h| \le P$ we have by (2.1) that

$$|3h\alpha_3 r - b_3| < 3C_3\gamma^{-8}P^{-2} < \tfrac{1}{64}P^{-1}$$

for P sufficiently large. Furthermore, we have

$$|S(h)| > C\gamma^{1/2}P > r^{1/2}P^{\varepsilon/6},$$

on choosing ε sufficiently small. Therefore, we may apply Baker's final coefficient lemma ([2], Lemma 4.6) to obtain an approximation to the coefficient of the linear term in (2.2). Writing $d = (r, b_3)$, we can find a positive integer $t \le 8$ such that

$$td^{-1}|3h\alpha_3 r - b_3| \le C^{-2}\gamma^{-1}P^{-2+\varepsilon} \tag{2.5}$$

and

$$||trd^{-1}(3h^2\alpha_3 + 2h\alpha_2)|| \le C^{-2}\gamma^{-1}P^{-1+\varepsilon}. \tag{2.6}$$

We note for future reference that $D|d$ and $D \le d \le 3D$.

Now write $x = x(h) = thD^{-1}$ and $\theta = 2r\alpha_2$. On writing z for the nearest integer to $x\theta$, we find using (2.5) and (2.6) that

$$\begin{aligned} |x\theta - z| &\le dD^{-1}||2trd^{-1}h\alpha_2|| \\ &\le 3||trd^{-1}(3h^2\alpha_3 + 2h\alpha_2)|| + 3htd^{-1}||3h\alpha_3 r|| \le 6C^{-2}\gamma^{-1}P^{-1+\varepsilon}. \end{aligned}$$

Put $X = 8P$ and $\zeta = 6C^{-2}\gamma^{-1}P^{-1+\varepsilon}$. As h runs through the set $\mathcal{B}$, the value of D is fixed, and there are only 8 possible values for t, so (2.4) shows that we generate R distinct, non-zero values of x, where

$$R \ge \tfrac{1}{8}\mathrm{card}(\mathcal{B}) \ge \tfrac{1}{4}C\gamma^{1/2}P. \tag{2.7}$$

Then for P sufficiently large we have $R > 24\zeta X$, so by Lemma 14 of Birch and Davenport [5] (see also Baker [2], Lemma 5.2) it must be the case that the ratio z/x is constant as h runs through $\mathcal{B}$. Therefore, we can find integers u and v, independent of h, with $(u, v) = 1$, such that $z/x = u/v$ for all values of x and corresponding values of z. Further, we can ensure that v is positive. Since u and v are coprime, we must have $v|x$ for all x. But

$$\tfrac{1}{5}\gamma^{1/4}PD^{-1} \le |x| \le 8PD^{-1}, \tag{2.8}$$

so it follows that $R \le 16P(vD)^{-1}$ and hence by (2.7) we see that

$$vD \le 64C^{-1}\gamma^{-1/2}. \tag{2.9}$$

Now for all $h \in \mathcal{B}$ we have by (2.7), (2.8), and (2.9) that

$$|v\theta - u| = v|x|^{-1}|x\theta - z| \le 5vD\gamma^{-1/4}P^{-1}\zeta \le 1920C^{-3}\gamma^{-7/4}P^{-2+\varepsilon}.$$

Finally, we set $q = 2vr$, $a_2 = u$, and $a_3 = 2vb$. Then (2.1) and (2.9) give

$$q \le 2vDr \le c_0\gamma^{-64-\frac{1}{2}}, \tag{2.10}$$

where we have set $c_0 = 128C_0C^{-1}$. Furthermore, we have

$$|q\alpha_2 - a_2| = |v\theta - u| \le c_2\gamma^{-7/4}P^{-2+\varepsilon} \tag{2.11}$$

and

$$|q\alpha_3 - a_3| = 2v|r\alpha_3 - b| < 2C_3v\gamma^{-8}P^{-3} \le c_3\gamma^{-8-\frac{1}{2}}P^{-3}, \tag{2.12}$$

where $c_2 = 1920C^{-3}$ and $c_3 = 48C_3C^{-1}$. If $(q, a_3, a_2) \ne 1$, then we may divide out the common factor and still retain the inequalities (2.10), (2.11), and (2.12) above. The lemma therefore follows on recalling that $\gamma \le 1$. □

We remark that Lemma 2.1 sharpens Baker's theorem only in cases where $|F(\boldsymbol{\alpha})|$ is nearly of order P, which is the situation that arises in our application. Baker's original theorem gives good results for sums as small as $P^{3/4+\varepsilon}$, this fact having been thoroughly exploited in our earlier paper [11].

3. The Davenport-Heilbronn Method

We now recall the analytic set-up introduced in [11]. We may assume (after rearranging variables) that the first m of the μ_i are zero, that the last n of the λ_i are zero, and that the remaining $h = s - m - n$ indices have both λ_i and μ_i nonzero. Then when $s \geq 13$ we have by conditions (i) and (ii) of Theorem 1 that

$$0 \leq m \leq 5, \quad 0 \leq n \leq 4, \quad \text{and} \quad h \geq 4. \tag{3.1}$$

Furthermore, we may suppose that λ_I/λ_J and μ_I/μ_J are irrational, where

$$I = m + h - 2 \quad \text{and} \quad J = m + h - 1.$$

We may also assume that $\tau = 1$, since this case may then be applied to the forms $\tau^{-1}F(\mathbf{x})$ and $\tau^{-1}G(\mathbf{x})$ to deduce the general result. When P and R are positive numbers, let

$$\mathcal{A}(P, R) = \{n \in [1, P] \cap \mathbb{Z} : p|n,\ p \text{ prime} \Rightarrow p \leq R\}$$

denote the set of R-smooth numbers up to P. Write $\boldsymbol{\alpha} = (\alpha_3, \alpha_2)$, and define generating functions

$$F_i(\boldsymbol{\alpha}) = F_i(\boldsymbol{\alpha}; P) = \sum_{1 \leq x \leq P} e(\lambda_i \alpha_3 x^3 + \mu_i \alpha_2 x^2)$$

and

$$f_i(\boldsymbol{\alpha}) = f_i(\boldsymbol{\alpha}; P, R) = \sum_{x \in \mathcal{A}(P,R)} e(\lambda_i \alpha_3 x^3 + \mu_i \alpha_2 x^2).$$

It will also be convenient to write

$$g_i(\alpha_3) = f_i(\alpha_3, 0) \quad \text{and} \quad H_i(\alpha_2) = F_i(0, \alpha_2).$$

Further, we set $R = P^\eta$. From now on, whenever a statement involves ε and R, it is intended to mean that the statement holds for all $\varepsilon > 0$, provided that η is sufficiently small in terms of ε. Finally, we assume throughout that P is chosen to be sufficiently large.

According to Davenport [8], there exists a real-valued even kernel function K of one real variable such that

$$K(\alpha) \ll \min(1, |\alpha|^{-2}) \tag{3.2}$$

and

$$\hat{K}(t) = \int_{-\infty}^{\infty} e(\alpha t) K(\alpha) d\alpha \begin{cases} = 0, & \text{if } |t| \geq 1, \\ \in [0, 1], & \text{if } |t| \leq 1, \\ = 1, & \text{if } |t| \leq \frac{1}{3}. \end{cases} \tag{3.3}$$

for all real numbers t. We set

$$\mathcal{K}(\boldsymbol{\alpha}) = K(\alpha_3) K(\alpha_2).$$

Now let $N(P)$ be the number of solutions of the system (1.1) satisfying

$$x_i \in \mathcal{A}(P, R) \qquad (i = 1, \ldots, m + h - 3)$$

and

$$1 \leq x_i \leq P \qquad (i = m + h - 2, \ldots, s).$$

By using (3.3), one finds that

$$N(P) \geq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})\mathcal{H}(\boldsymbol{\alpha})\mathcal{K}(\boldsymbol{\alpha})\,d\boldsymbol{\alpha}, \tag{3.4}$$

where

$$\mathcal{F}(\boldsymbol{\alpha}) = \prod_{i=1}^{m+h-3} f_i(\boldsymbol{\alpha}), \quad \mathcal{H}(\boldsymbol{\alpha}) = \prod_{i=m+h-2}^{m+h} F_i(\boldsymbol{\alpha}), \quad \text{and} \quad \mathcal{G}(\boldsymbol{\alpha}) = \prod_{i=m+h+1}^{s} F_i(\boldsymbol{\alpha}).$$

Before describing the dissection of the plane into major, minor, and trivial arcs, we need the following two lemmas, which are straightforward extensions of the ideas of Freeman [10], Lemmas 3 and 4.

Lemma 3.1. *Suppose that S_j and T_j are fixed real numbers satisfying*

$$0 < S_j \leq 1 \leq T_j \qquad (j = 2, 3).$$

Then whenever $\{i, j\} = \{2, 3\}$, one has

$$\lim_{P\to\infty}\left(\sup_{\substack{S_j \leq |\alpha_j| \leq T_j \\ \alpha_i \in \mathbb{R}}} \frac{|F_I(\boldsymbol{\alpha}; P)F_J(\boldsymbol{\alpha}; P)|}{P^2}\right) = 0.$$

Proof. Let us first suppose that $i = 2$ and $j = 3$. For notational convenience, we write $(\alpha_3, \alpha_2) = (\alpha, \beta)$. If the result fails to hold, then we can find $\varepsilon > 0$, a sequence of positive real numbers $\{P_n\}$ tending to ∞, and a sequence of ordered pairs $\{\boldsymbol{\alpha}_n\} = \{(\alpha_n, \beta_n)\}$ with

$$S_3 \leq |\alpha_n| \leq T_3 \quad \text{and} \quad \beta_n \in \mathbb{R} \quad (n \in \mathbb{Z}^+),$$

such that

$$|F_I(\boldsymbol{\alpha}_n; P_n)F_J(\boldsymbol{\alpha}_n; P_n)| \geq \varepsilon P_n^2$$

for all positive integers n. On making a trivial estimate, it follows that for each n one has

$$|F_i(\boldsymbol{\alpha}_n; P_n)| \geq \varepsilon P_n \quad (i = I, J).$$

Whenever n is large enough so that $P_n \geq \varepsilon^{-512}$, we may apply Lemma 2.1 with $\gamma = \varepsilon^8$. Thus we obtain integers q_{in} and a_{in} satisfying

$$q_{in} \leq c_0\varepsilon^{-520} \quad \text{and} \quad |\lambda_i\alpha_n q_{in} - a_{in}| \leq c_3\varepsilon^{-72}P_n^{-3} \quad (i = I, J). \tag{3.5}$$

It follows that for n sufficiently large one has

$$a_{in} \leq c_0|\lambda_i|T_3\varepsilon^{-520} + c_3\varepsilon^{-72} \ll 1,$$

and hence there are only finitely many possible 4-tuples $(a_{In}, q_{In}, a_{Jn}, q_{Jn})$. So there must be a 4-tuple (a_I, q_I, a_J, q_J) that occurs for infinitely many of the α_n. The compactness of $[S_3, T_3]$ then ensures that among these α_n we can find a subsequence $\{\alpha_{n_\ell}\}$ converging to a non-zero limit α_0. We have

$$|\lambda_I\alpha_{n_\ell}q_I - a_I| \leq c_3\varepsilon^{-72}P_{n_\ell}^{-3} \quad \text{and} \quad |\lambda_J\alpha_{n_\ell}q_J - a_J| \leq c_3\varepsilon^{-72}P_{n_\ell}^{-3}$$

for each ℓ, so on letting $\ell \to \infty$ we find that $\lambda_I/\lambda_J = a_Iq_J/(a_Jq_I)$, contradicting the assumption that λ_I/λ_J is irrational.

For the case where $i = 3$ and $j = 2$, we repeat the same argument except that in (3.5) we use the second inequality of Lemma 2.1 instead of the third and eventually contradict the irrationality of μ_I/μ_J. □

Lemma 3.2. *There exist positive, real-valued functions $S_j(P)$ and $T_j(P)$, depending only on $\lambda_I, \lambda_J, \mu_I$, and μ_J, such that*

$$\lim_{P\to\infty} S_j(P) = 0 \quad \textit{and} \quad \lim_{P\to\infty} T_j(P) = \infty \quad (j = 2, 3)$$

and whenever $\{i, j\} = \{2, 3\}$ one has

$$\lim_{P\to\infty} \left(\sup_{\substack{S_j(P)\le|\alpha_j|\le T_j(P)\\ \alpha_i\in\mathbb{R}}} \frac{|F_I(\boldsymbol{\alpha}; P)F_J(\boldsymbol{\alpha}; P)|}{P^2} \right) = 0.$$

Proof. Fix $j = 2$ or 3. Then for every positive integer m, Lemma 3.1 tells us that there is a real number $P_m = P_{m,j}$ such that

$$\frac{|F_I(\boldsymbol{\alpha}; P)F_J(\boldsymbol{\alpha}; P)|}{P^2} \le \frac{1}{m} \quad \text{whenever} \quad P \ge P_m \quad \text{and} \quad \frac{1}{m} \le |\alpha_j| \le m,$$

and we may clearly assume that the sequence $\{P_m\}$ is non-decreasing. Now when P satisfies $P_m \le P < P_{m+1}$, we define

$$S_j(P) = \frac{1}{m} \quad \text{and} \quad T_j(P) = m.$$

It follows easily that

$$\frac{|F_I(\boldsymbol{\alpha}; P)F_J(\boldsymbol{\alpha}; P)|}{P^2} \le \frac{1}{m} \quad \text{when} \quad P \ge P_m \quad \text{and} \quad S_j(P) \le |\alpha_j| \le T_j(P),$$

and this suffices to complete the proof. □

We are now ready to describe the dissection of the plane that will be used to evaluate the integral (3.4). Let $T_j(P)$ be as in Lemma 3.2, and define the trivial arcs by

$$\mathfrak{t} = \{\boldsymbol{\alpha} : |\alpha_3| > T_3(P) \ \text{ or } \ |\alpha_2| > T_2(P)\}.$$

Write

$$\lambda = 18 \max_{1\le i\le s} |\lambda_i| \quad \text{and} \quad \mu = 18 \max_{1\le i\le s} |\mu_i|,$$

and define

$$\mathfrak{M} = \{\boldsymbol{\alpha} : |\alpha_3| \le \lambda^{-1}P^{-2} \text{ and } |\alpha_2| \le \mu^{-1}P^{-1}\}$$

to be the major arc. Finally, the minor arcs are given by

$$\mathfrak{m} = \mathbb{R}^2 \setminus (\mathfrak{t} \cup \mathfrak{M}).$$

As in [11] and [13], we have for any set $\mathfrak{n} \in \mathbb{R}^2$ that

$$\int_{\mathfrak{n}} |\mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})|d\boldsymbol{\alpha} \ll \int_{\mathfrak{n}} |f_i(\boldsymbol{\alpha})|^{h-3}|g_j(\alpha_3)|^m|H_k(\alpha_2)|^n d\boldsymbol{\alpha} \tag{3.6}$$

for some i, j, and k satisfying

$$m + 1 \le i \le m + h, \quad 1 \le j \le m, \quad \text{and} \quad m + h + 1 \le k \le s. \tag{3.7}$$

With the abbreviations

$$f = |f_i(\boldsymbol{\alpha})|, \quad g = |g_j(\alpha_3)|, \quad \text{and} \quad H = |H_k(\alpha_2)|,$$

we find using (3.1) that

$$f^{h-3} g^m H^n \ll P^{s-13} \left(f^{10} + f^6 H^4 + g^6 H^4 + f^4 g^6 \right). \tag{3.8}$$

In order to employ this decomposition, we need several mean value estimates. In each one, we are required to obtain the full savings of P^5 with fewer than 13 variables present, as the minor-arc bounds for F_I and F_J stemming from Lemma 3.2 do not provide quantifiable gains over the trivial estimates.

Lemma 3.3. *Suppose that i, j, and k satisfy* (3.7) *and that $m + h - 2 \le \ell \le m + h$. Then for any $t > 8/3$ and any unit square $\mathcal{U} = [c, c+1] \times [d, d+1]$, one has*

(i) $\displaystyle\int_{\mathcal{U}} |F_\ell(\boldsymbol{\alpha})|^t |f_i(\boldsymbol{\alpha})|^{10} d\boldsymbol{\alpha} \ll P^{t+5}$,

(ii) $\displaystyle\int_{\mathcal{U}} |F_\ell(\boldsymbol{\alpha})|^t |f_i(\boldsymbol{\alpha})|^6 |H_k(\alpha_2)|^4 d\boldsymbol{\alpha} \ll P^{t+5}$,

(iii) $\displaystyle\int_{\mathcal{U}} |F_\ell(\boldsymbol{\alpha})|^t |g_j(\alpha_3)|^6 |H_k(\alpha_2)|^4 d\boldsymbol{\alpha} \ll P^{t+5}$,

(iv) $\displaystyle\int_{\mathcal{U}} |F_\ell(\boldsymbol{\alpha})|^t |f_i(\boldsymbol{\alpha})|^4 |g_j(\alpha_3)|^6 d\boldsymbol{\alpha} \ll P^{t+5}$.

Proof. We dissect $\mathcal{U}$ into major and minor arcs as follows. Let

$$\mathcal{M}(q, a, b) = \{ \boldsymbol{\alpha} \in \mathcal{U} : |\lambda_\ell \alpha_3 q - a| < P^{-9/4} \text{ and } |\mu_\ell \alpha_2 q - b| < P^{-5/4} \},$$

and write

$$\mathcal{M} = \bigcup_{\substack{0 \le a, b \le q < P^{3/4} \\ (q,a,b)=1}} \mathcal{M}(q, a, b).$$

Then by Baker [2], Theorem 5.1, one has $|F_\ell(\boldsymbol{\alpha})| \ll P^{3/4+\varepsilon}$ whenever $\boldsymbol{\alpha} \in \mathcal{U} \setminus \mathcal{M}$. Therefore, by part (i) of [11], Lemma 5 (see also [14], Theorem 2), one has

$$\int_{\mathcal{U} \setminus \mathcal{M}} |F_\ell(\boldsymbol{\alpha})|^t |f_i(\boldsymbol{\alpha})|^{10} d\boldsymbol{\alpha} \ll P^{(3/4+\varepsilon)t} \cdot P^{17/3+\varepsilon} \ll P^{t+5}$$

for ε sufficiently small, since $t > 8/3$. Similar minor arc bounds for the integrals in (ii)–(iv) follow by using parts (ii)–(iv) of [11], Lemma 5.

For the major arcs, we again illustrate the argument by focusing attention on the integral in part (i). By Hölder's inequality, one has

$$\int_{\mathcal{M}} |F_\ell(\boldsymbol{\alpha})|^t |f_i(\boldsymbol{\alpha})|^{10} d\boldsymbol{\alpha} \le \left(\int_{\mathcal{M}} |F_\ell(\boldsymbol{\alpha})|^{3t} d\boldsymbol{\alpha} \right)^{1/3} \left(\int_{\mathcal{U}} |f_i(\boldsymbol{\alpha})|^{15} d\boldsymbol{\alpha} \right)^{2/3},$$

and the result now follows on making a change of variables and using [11], Lemma 8, together with part (v) of [11], Lemma 5. Estimates for the major arc integrals in (ii)–(iv) follow in an identical manner on using parts (vi)–(viii) of [11], Lemma 5. □

The trivial arcs are now quite easy to handle. Since

$$|\mathcal{H}(\boldsymbol{\alpha})| \leq |F_I(\boldsymbol{\alpha})|^3 + |F_J(\boldsymbol{\alpha})|^3 + |F_K(\boldsymbol{\alpha})|^3, \tag{3.9}$$

where $K = m + h$, we find from (3.2), (3.6), (3.8), and Lemma 3.3 that

$$\int_{\mathfrak{t}} |\mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})\mathcal{H}(\boldsymbol{\alpha})\mathcal{K}(\boldsymbol{\alpha})|\, d\boldsymbol{\alpha} \ll (T_2(P)^{-1} + T_3(P)^{-1})P^{s-5},$$

and since $T_j(P) \to \infty$ we see that this is $o(P^{s-5})$.

Let us now tackle the minor arcs. We first subdivide $\mathfrak{m}$ into two regions. Let $S_j(P) \geq P^{-1}$ be as in Lemma 3.2, and put

$$\mathfrak{m}_1 = \{\boldsymbol{\alpha} \in \mathfrak{m} : |\alpha_3| \geq S_3(P) \ \text{ or } \ |\alpha_2| \geq S_2(P)\}$$

and $\mathfrak{m}_2 = \mathfrak{m} \setminus \mathfrak{m}_1$. We know from Lemma 3.2 that

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{m}_1} |F_I(\boldsymbol{\alpha})F_J(\boldsymbol{\alpha})| = o(P^2). \tag{3.10}$$

Now we need a similar result on the set $\mathfrak{m}_2$. The basic idea is that if $\boldsymbol{\alpha} \in \mathfrak{m}_2$ and $|F_I(\boldsymbol{\alpha})|$ is large, then $\lambda_I\alpha_3$ and $\mu_I\alpha_2$ have good rational approximations, yet both are already close to zero when P is large, since $S_j(P) \to 0$. We may therefore hope to get a contradiction by showing that $\boldsymbol{\alpha}$ must then in fact lie in the major arc. Suppose that $\boldsymbol{\alpha} \in \mathfrak{m}_2$ and that $|F_I(\boldsymbol{\alpha})| \geq \gamma^{1/8}P$, where

$$\gamma = (\max\{S_2(P), S_3(P)\})^{1/66}.$$

Since $S_j(P) \geq P^{-1}$ we have $\gamma \geq P^{-1/66}$, and hence Lemma 2.1 applies. Thus we obtain integers q, a_2, and a_3, with $(q, a_2, a_3) = 1$, such that

$$1 \leq q \leq c_0\gamma^{-65}, \quad |\mu_I\alpha_2 q - a_2| \leq c_2\gamma^{-2}P^{-2+\varepsilon}, \quad \text{and} \quad |\lambda_I\alpha_3 q - a_3| \leq c_3\gamma^{-9}P^{-3}.$$

It follows that

$$|a_3| \leq c_3\gamma^{-9}P^{-3} + |\lambda_I\alpha_3|q \ll \gamma^{-9}P^{-3} + \gamma^{-65}S_3(P) \ll P^{-2} + S_3(P)^{1/66},$$

and similarly

$$|a_2| \leq c_2\gamma^{-2}P^{-2+\varepsilon} + |\mu_I\alpha_2|q \ll \gamma^{-2}P^{-2+\varepsilon} + \gamma^{-65}S_2(P) \ll P^{-1} + S_2(P)^{1/66},$$

whence $a_2 = a_3 = 0$ when P is sufficiently large. Therefore we have $|\alpha_3| \ll \gamma^{-9}P^{-3}$ and $|\alpha_2| \ll \gamma^{-2}P^{-2+\varepsilon}$. For sufficiently large P, this implies that $\boldsymbol{\alpha} \in \mathfrak{M}$ and hence gives a contradiction. We therefore conclude that

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{m}_2} |F_I(\boldsymbol{\alpha})| \leq \gamma^{1/8}P = o(P). \tag{3.11}$$

Now we are ready to complete the minor arc analysis. By (3.2) and (3.9), we have for some ℓ with $m + h - 2 \leq \ell \leq m + h$ and some unit square $\mathcal{U}$ that

$$\int_{\mathfrak{m}} |\mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})\mathcal{H}(\boldsymbol{\alpha})\mathcal{K}(\boldsymbol{\alpha})|\, d\boldsymbol{\alpha} \ll \sup_{\boldsymbol{\alpha} \in \mathfrak{m}} |F_I(\boldsymbol{\alpha})F_J(\boldsymbol{\alpha})|^{1/8} \int_{\mathcal{U}} |F_\ell(\boldsymbol{\alpha})|^{11/4} |\mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})|\, d\boldsymbol{\alpha}.$$

Therefore by (3.6), (3.8), (3.10), (3.11), and Lemma 3.3 we have

$$\int_{\mathfrak{m}} |\mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})\mathcal{H}(\boldsymbol{\alpha})\mathcal{K}(\boldsymbol{\alpha})|\, d\boldsymbol{\alpha} \ll \sup_{\boldsymbol{\alpha} \in \mathfrak{m}} |F_I(\boldsymbol{\alpha})F_J(\boldsymbol{\alpha})|^{1/8} P^{s-21/4} = o(P^{s-5}).$$

The treatment of the major arc is almost identical to that of [11], so our discussion will be somewhat brief. As usual, we must prune back to a smaller set $\mathfrak{N}$ on which we can obtain asymptotics for the sums $f_i(\boldsymbol{\alpha})$. We let $W = (\log P)^{1/4}$ and define

$$\mathfrak{N} = \{\boldsymbol{\alpha} : |\alpha_3| \le WP^{-3} \text{ and } |\alpha_2| \le WP^{-2}\}.$$

Then by using Hölder's inequality, together with Lemma 9.2 of Wooley [13] and Lemma 3.3, we find that

$$\int_{\mathfrak{M}\setminus\mathfrak{N}} |\mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})\mathcal{H}(\boldsymbol{\alpha})\mathcal{K}(\boldsymbol{\alpha})|d\boldsymbol{\alpha} \ll P^{s-5}W^{-\sigma}$$

for some $\sigma > 0$. It may be worth mentioning that Freeman [10] is able to avoid pruning entirely in his work on a single inequality. The factor of P^ε in our estimate for $||q\alpha_2||$ in Lemma 2.1 is what prevents us from extending the $\mathfrak{m}_2$ analysis down to the boundary of $\mathfrak{N}$ in the α_2 direction.

When $\boldsymbol{\alpha} \in \mathfrak{N}$, we are able to approximate $F_i(\boldsymbol{\alpha})$ and $f_i(\boldsymbol{\alpha})$ by the functions

$$v_i(\boldsymbol{\alpha}) = \int_0^P e(\lambda_i\alpha_3\gamma^3 + \mu_i\alpha_2\gamma^2)\,d\gamma$$

and

$$w_i(\boldsymbol{\alpha}) = \int_R^P \rho\left(\frac{\log\gamma}{\log R}\right) e(\lambda_i\alpha_3\gamma^3 + \mu_i\alpha_2\gamma^2)\,d\gamma$$

as in [11]. Here $\rho(x)$ denotes Dickman's function (see for example Vaughan [12], chapter 12). Thus we are able to show that

$$\int_{\mathfrak{N}} \mathcal{F}(\boldsymbol{\alpha})\mathcal{G}(\boldsymbol{\alpha})\mathcal{H}(\boldsymbol{\alpha})\mathcal{K}(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} \sim J(P),$$

where

$$J(P) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\prod_{i=1}^{m+h-3} w_i(\boldsymbol{\alpha})\right)\left(\prod_{i=m+h-2}^{s} v_i(\boldsymbol{\alpha})\right)\mathcal{K}(\boldsymbol{\alpha})\,d\boldsymbol{\alpha}$$

denotes the singular integral. By arguing as in [11], we find that

$$J(P) \gg P^s \int_{\mathcal{B}} \hat{K}(F(\boldsymbol{\gamma})P^3)\hat{K}(G(\boldsymbol{\gamma})P^2)\,d\boldsymbol{\gamma},$$

where $\mathcal{B} = [R/P, 1]^{m+h-3} \times [0,1]^{n+3}$. Now by condition (iii) of Theorem 1 and the argument of Lemma 6.2 of Wooley [13], we may assume that there is a non-singular solution $\boldsymbol{\eta}$ to the equations $F = G = 0$ such that $\boldsymbol{\eta}$ lies in the interior of $\mathcal{B}$ when P is sufficiently large. By the inverse function theorem, we are then able to find a set $V \in \mathbb{R}^2$ containing the origin, with $\text{meas}(V) \gg 1$, such that

$$J(P) \gg P^s \int_V \hat{K}(z_jP^3)\hat{K}(z_kP^2)\,d\mathbf{z}.$$

It now follows from (3.3) that $J(P) \gg P^{s-5}$, and this completes the proof of Theorem 1.

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