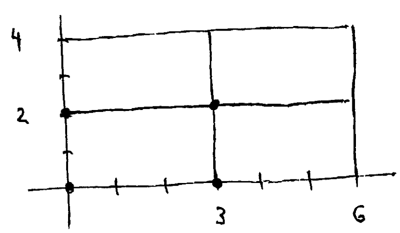
# MAT 261—Exam #3—4/17/14

Name: Solutions

Calculators are not permitted. Show all of your work using correct mathematical notation.

1. (15 points) Consider the integral $\int_0^6 \int_0^4 (x + 2y)\, dy\, dx$.

   (a) Find the Riemann sum approximation $S_{2,2}$ to the integral, using 4 rectangles with $\Delta x = 3$ and $\Delta y = 2$ and the lower left vertices as sample points.




$$S_{2,2} = \Delta A \left( f(0,0) + f(3,0) + f(0,2) + f(3,2) \right)$$
$$= 6\,(0 + 3 + 4 + 7)$$
$$= 84$$

$$f(x,y) = x + 2y$$

$$\Delta A = \Delta x\, \Delta y$$
$$= 3 \cdot 2 = 6$$

   (b) Find the exact value of the integral.

$$\int_0^6 \int_0^4 (x + 2y)\, dy\, dx = \int_0^6 (xy + y^2) \Big|_0^4 dx$$
$$= \int_0^6 (4x + 16)\, dx$$
$$= 2x^2 + 16x \Big|_0^6$$
$$= 72 + 96$$
$$= 168$$

2. (15 points) Find the average value of the function $f(x, y, z) = \dfrac{e^{2z}}{(x+3y)^2}$ over the box $[1, 5] \times [0, 2] \times [0, 1]$.

$$\bar{f} = \frac{1}{8}\int_1^5\int_0^2\int_0^1 e^{2z}(x+3y)^{-2}\,dz\,dy\,dx$$

$$= \frac{1}{8}\int_1^5\int_0^2 (x+3y)^{-2}\cdot\frac{1}{2}e^{2z}\Big|_0^1\,dy\,dx$$

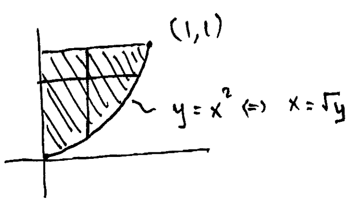
$$= \frac{e^2-1}{16}\int_1^5 -\frac{1}{3}(x+3y)^{-1}\Big|_0^2\,dx$$

$$= \frac{e^2-1}{48}\int_1^5\left(\frac{1}{x}-\frac{1}{x+6}\right)dx$$

$$= \frac{e^2-1}{48}\left(\ln|x| - \ln|x+6|\right)\Big|_1^5$$

$$= \frac{e^2-1}{48}\left(\ln 5 - \ln 11 + \ln 7\right) = \frac{e^2-1}{48}\ln\left(35/11\right)$$

3. (15 points) Evaluate the integral $\displaystyle\int_0^1\int_{x^2}^1 x^3\sin(\pi y^3)\,dy\,dx$ by reversing the order of integration. Include a sketch of the domain.




$$\int_0^1\int_0^{\sqrt{y}} x^3\sin(\pi y^3)\,dx\,dy$$

$$= \int_0^1 \frac{x^4}{4}\sin(\pi y^3)\Big|_{x=0}^{\sqrt{y}}\,dy$$

$$= \int_0^1 \frac{y^2}{4}\sin(\pi y^3)\,dy \qquad u = \pi y^3,\quad du = 3\pi y^2\,dy$$

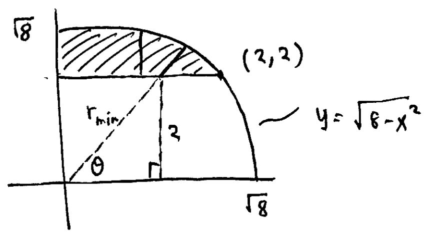
$$= \frac{1}{12\pi}\int_0^{\pi}\sin u\,du$$

$$= -\frac{1}{12\pi}\cos u\Big|_0^{\pi}$$

$$= -\frac{1}{12\pi}(-1-1)$$

$$= \frac{1}{6\pi}$$

4. (15 points) Evaluate the integral $\int_0^2 \int_2^{\sqrt{8-x^2}} (x^2+y^2)^{-3/2}\,dy\,dx$ by changing to polar coordinates. Include a sketch of the domain.




$$\sin\theta = \frac{2}{r_{min}}$$

$$\Rightarrow\quad r_{min} = \frac{2}{\sin\theta}$$

At $(2,2)$, $\theta = \pi/4$

$$\int_{\pi/4}^{\pi/2}\int_{2/\sin\theta}^{\sqrt8} r^{-3}\cdot r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} -\frac{1}{r}\Big|_{2/\sin\theta}^{\sqrt8}\,d\theta$$

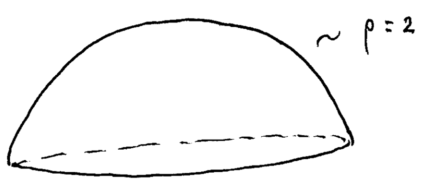
$$= \int_{\pi/4}^{\pi/2}\left(\frac12\sin\theta - \frac{1}{\sqrt8}\right)d\theta$$

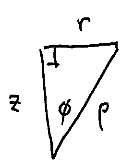
$$= \frac12\left(-\cos\theta - \frac{1}{\sqrt2}\theta\right)\Big|_{\pi/4}^{\pi/2}$$

$$= \frac12\left(\frac{1}{\sqrt2} - \frac{1}{\sqrt2}\cdot\frac{\pi}{4}\right)$$

$$= \frac{4-\pi}{8\sqrt2}$$

5. (15 points) An object occupying the hemisphere defined by $x^2+y^2+z^2 \leqslant 4$ and $z \geqslant 0$ has mass density $\delta(x,y,z) = 3z^2$ kg per cubic unit. Find the total mass of the object.




$$z = \rho\cos\phi$$

$$M = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} 3(\rho\cos\phi)^2\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 6\pi\int_0^{\pi/2}\cos^2\phi\,\sin\phi\,d\phi\int_0^2 \rho^4\,d\rho$$

$u = \cos\phi$
$du = -\sin\phi\,d\phi$

$$= 6\pi\cdot\frac{2^5}{5}\int_0^1 u^2\,du$$

$$= 6\pi\cdot\frac{32}{5}\cdot\frac13$$

$$= \frac{64\pi}{5}$$

Alternatively, one could use cylindrical coordinates:

$$M = \int^{2\pi}\int^{2}\int^{\sqrt{4-r^2}} 3z^2\,r\,dz\,dr\,d\theta$$

6. (15 points) Consider the integral $\iint_{\mathcal{D}} (y-x)^5 \, dA$, where $\mathcal{D}$ is the parallelogram in the $xy$-plane spanned by the vectors $\langle 4,5\rangle$ and $\langle 1,3\rangle$. Use the transformation

$$\Phi(u,v) = (4u+v, 5u+3v)$$

to evaluate the integral.

$\Phi$ maps $[0,1]^2$ to $\mathcal{D}$, and $\mathrm{Jac}(\Phi) = \begin{vmatrix} 4 & 1 \\ 5 & 3 \end{vmatrix} = 7$,

so
$$\iint_{\mathcal{D}} (y-x)^5 \, dA = 7\int_0^1\int_0^1 (u+2v)^5 \, dv \, du$$
$$= \frac{7}{12}\int_0^1 (u+2v)^6 \Big|_{v=0}^{1} du$$
$$= \frac{7}{12}\int_0^1 \left[(u+2)^6 - u^6\right] du$$
$$= \frac{7}{12}\cdot\frac{1}{7}\left[(u+2)^7 - u^7\right]_0^1$$
$$= \frac{1}{12}\left(3^7 - 2^7 - 1\right)$$

7. (10 points) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{s}$ for the vector field $\mathbf{F} = \langle x^2, xy\rangle$ and the curve $C$ parametrized by $\mathbf{c}(t) = \langle t^3, 2t\rangle$ on the interval $0 \leqslant t \leqslant 1$.

$$\vec{F}(\vec{c}(t)) = \langle t^6, 2t^4 \rangle$$
$$\vec{c}\,'(t) = \langle 3t^2, 2 \rangle$$

$$\int_C \vec{F}\cdot d\vec{s} = \int_0^1 \vec{F}(\vec{c}(t))\cdot\vec{c}\,'(t)\, dt$$
$$= \int_0^1 (3t^8 + 4t^4)\, dt$$
$$= \frac{1}{3}t^9 + \frac{4}{5}t^5 \Big|_0^1$$
$$= \frac{1}{3} + \frac{4}{5}$$
$$= \frac{17}{15}$$