MAT 261—Exam #1A—2/18/14

Name: Solutions

Calculators are not permitted. Show all work using correct mathematical notation.

1. (10 points) Find the slope of the tangent line to the curve $c(t) = (4\ln t, t^3 + 5)$ at the point where $t = 2$. Give your answer in simplest possible form.

$x(t) = 4\ln t$ $\qquad$ $y(t) = t^3 + 5$

$\Rightarrow x'(t) = \frac{4}{t}$ $\qquad$ $\Rightarrow y'(t) = 3t^2$

$\Rightarrow x'(2) = 2$ $\qquad$ $\Rightarrow y'(2) = 12$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{y'(2)}{x'(2)} = 6$$

2. (15 points) A particle moves in space with trajectory

$$x(t) = e^{3t}, \quad y(t) = t^2 \sin \pi t, \quad z(t) = \frac{4}{(t+1)^2}.$$

Find the speed of the particle at $t = 1$. Give your answer in simplest possible form.

$x'(t) = 3e^{3t} \Rightarrow x'(1) = 3e^3$

$y'(t) = t^2 \cdot \pi \cos \pi t + (\sin \pi t) \cdot 2t$

$\Rightarrow y'(1) = -\pi$

$z'(t) = -8(t+1)^{-3} \Rightarrow z'(1) = -1$

$$\left.\frac{ds}{dt}\right|_{t=1} = \sqrt{(3e^3)^2 + (-\pi)^2 + (-1)^2}$$

$$= \sqrt{9e^6 + \pi^2 + 1}$$

3. (13 points) Find the angle between the vectors $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. You may express your answer in terms of inverse trigonometric functions.

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$$

$$= \frac{1 - 6 + 1}{\sqrt{11} \cdot \sqrt{6}}$$

$$= \frac{-4}{\sqrt{66}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{-4}{\sqrt{66}}\right)$$

4. (12 points) Find the area of the parallelogram determined by the vectors $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \hat{i} - \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \hat{j} + \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \hat{k}$$

$$= \hat{i} - 5\hat{j} + 2\hat{k}$$

$$\text{Area} = \|\vec{v} \times \vec{w}\|$$

$$= \sqrt{1^2 + (-5)^2 + 2^2} = \sqrt{30}$$

5. (10 points) Find parametric equations for the line that is perpendicular to the plane $x + 2y + 4z = 8$ and passes through the point $(7, 5, 3)$.

$\vec{n} = \langle 1, 2, 4 \rangle$ is a normal vector to the plane and hence a direction vector for the line.

So the line has the vector equation

$$\langle x, y, z \rangle = \langle 7, 5, 3 \rangle + \langle 1, 2, 4 \rangle t$$

and parametric equations

$$x = 7 + t, \quad y = 5 + 2t, \quad z = 3 + 4t.$$

6. (15 points) Find a unit vector perpendicular to the plane containing the points $P(2, -1, 0)$, $Q(1, 0, 1)$, and $R(0, 3, -1)$.

$\vec{PQ} = \langle -1, 1, 1 \rangle$ and $\vec{PR} = \langle -2, 4, -1 \rangle$ are vectors in the plane, so

$$\vec{v} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & 1 \\ -2 & 4 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} -1 & 1 \\ -2 & 4 \end{vmatrix} \hat{k}$$

$$= -5\hat{\imath} - 3\hat{\jmath} - 2\hat{k}$$

is perpendicular to the plane.

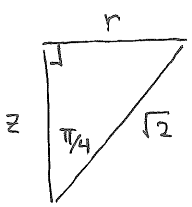
Hence $$\vec{e}_{\vec{v}} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{-1}{\sqrt{38}} \left(5\hat{\imath} + 3\hat{\jmath} + 2\hat{k} \right)$$

is a unit vector perpendicular to the plane.

Note that $-\vec{e}_{\vec{v}} = \frac{1}{\sqrt{38}} \left(5\hat{\imath} + 3\hat{\jmath} + 2\hat{k} \right)$

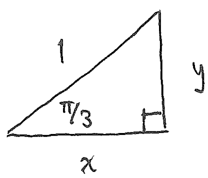
is also acceptable.

7. (10 points) Convert the spherical coordinates $(\rho, \theta, \phi) = (\sqrt{2}, \pi/3, \pi/4)$ into rectangular coordinates (x, y, z).



$$z = \sqrt{2} \cos \pi/4 = 1$$

$$r = \sqrt{2} \sin \pi/4 = 1$$



$$x = \cos \pi/3 = 1/2$$

$$y = \sin \pi/3 = \sqrt{3}/2$$

So $(x, y, z) = (1/2, \sqrt{3}/2, 1)$

8. (15 points) Find the length of the helix defined by

$$\mathbf{r}(t) = (\cos 5t)\mathbf{i} + (\sin 5t)\mathbf{j} + 2t^{3/2}\mathbf{k} \qquad (0 \leqslant t \leqslant 1).$$

$$\vec{r}'(t) = (-5 \sin 5t)\,\hat{\imath} + (5 \cos 5t)\,\hat{\jmath} + (3t^{1/2})\,\hat{k}$$

$$\Rightarrow \quad \|\vec{r}'(t)\| = \sqrt{25 \sin^2 5t + 25 \cos^2 5t + 9t}$$

$$= \sqrt{25 + 9t}$$

$$\therefore \quad s = \int_0^1 \sqrt{25 + 9t}\, dt \qquad \text{Let } u = 25 + 9t, \quad du = 9\, dt$$

$$= \frac{1}{9} \int_{25}^{34} \sqrt{u}\, du$$

$$= \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_{25}^{34}$$

$$= \frac{2}{27} \left(34^{3/2} - 125 \right)$$