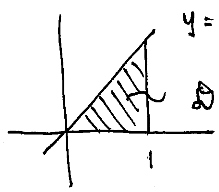
MAT 261—Exam #3—11/13/14

Name: Solutions

Calculators are not permitted. Show all of your work using correct mathematical notation.

1. (15 points) Find the average value of the function $f(x,y) = x + \sqrt{y}$ over the triangle bounded by the lines $y = 0$, $x = 1$, and $y = x$.



Area $(D) = \frac{1}{2}$

$$\bar{f} = 2\int_0^1 \int_0^x (x + \sqrt{y})\, dy\, dx$$

$$= 2\int_0^1 \left(xy + \tfrac{2}{3} y^{3/2} \right) \Big|_0^x \, dx$$

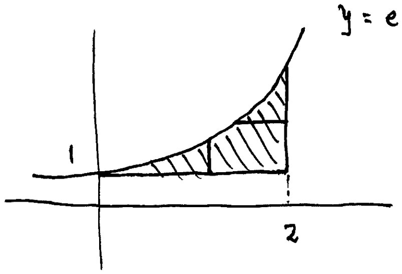
$$= 2\int_0^1 \left(x^2 + \tfrac{2}{3} x^{3/2} \right) dx$$

$$= 2\left(\frac{x^3}{3} + \frac{4}{15} x^{5/2} \right) \Big|_0^1$$

$$= 2\left(\frac{1}{3} + \frac{4}{15} \right)$$

$$= 6/5$$

2. (10 points) Consider the integral $\int_0^2 \int_1^{e^x} f(x,y)\, dy\, dx$. Sketch the domain of integration, and set up an equivalent integral with the order of integration reversed.

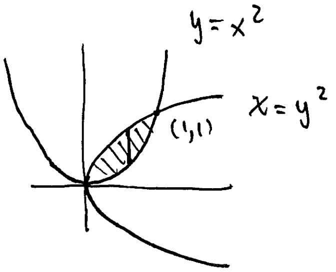


$y = e^x \leftrightarrow x = \ln y$

$$\int_0^2 \int_1^{e^x} f(x,y)\, dy\, dx$$

$$= \int_1^{e^2} \int_{\ln y}^{2} f(x,y)\, dx\, dy$$

3. (10 points) Set up (**but do not evaluate**) an integral that gives the volume of a solid whose base is the region in the xy-plane between the curves $y = x^2$ and $x = y^2$ and whose upper boundary is the elliptical paraboloid $z = 9 - x^2 - 2y^2$.



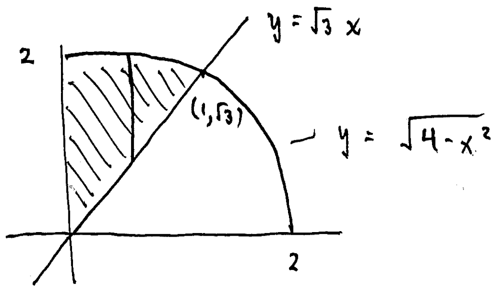
Base

Height function : $f(x,y) = 9 - x^2 - 2y^2$

$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} (9 - x^2 - 2y^2)\, dy\, dx$$

4. (15 points) Evaluate the integral $\displaystyle\int_0^1 \int_{\sqrt{3}x}^{\sqrt{4-x^2}} (x^4y + x^2y^3)\, dy\, dx$ by changing to polar coordinates. Include a sketch of the domain.

$x^2y(x^2+y^2)$



Along the line $y = \sqrt{3}x$ we have $\tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$, so the integral becomes

$$\int_{\pi/3}^{\pi/2} \int_0^2 (r\cos\theta)^2 (r\sin\theta)\, r^2 \cdot r\, dr\, d\theta$$

$$= \int_{\pi/3}^{\pi/2} \cos^2\theta \sin\theta\, d\theta \int_6^2 r^6\, dr$$

$$= -\frac{\cos^3\theta}{3}\Big|_{\pi/3}^{\pi/2} \cdot \frac{r^7}{7}\Big|_0^2$$

$$= \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 \cdot \frac{2^7}{7} = \frac{16}{21}$$

5. (15 points) Evaluate the triple integral $\int_0^1 \int_0^2 \int_0^1 \frac{yz^4 \sin(\pi x)}{3+y^2}\, dz\, dy\, dx$.

$$= \int_0^1 \sin(\pi x)\, dx \int_0^2 \frac{y}{3+y^2}\, dy \int_0^1 z^4\, dz$$

$$= -\frac{1}{\pi}\cos(\pi x)\Big|_0^1 \cdot \frac{1}{2}\ln(3+y^2)\Big|_0^2 \cdot \frac{1}{5}z^5\Big|_0^1$$

$$= -\frac{1}{\pi}(-1-1) \cdot \frac{1}{2}(\ln 7 - \ln 3) \cdot \frac{1}{5}$$

$$= \frac{1}{5\pi}\ln(7/3)$$

6. (15 points) Consider the integral $\iint_{\mathcal{D}} (x+y)\, dA$, where $\mathcal{D}$ is the parallelogram in the xy-plane spanned by the vectors $\langle 5, 2\rangle$ and $\langle 1, 3\rangle$. Use the transformation

$$G(u,v) = (5u+v, 2u+3v)$$

to evaluate the integral.

$$\text{Jac}(G) = \begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} = 13$$

$$\iint_{\mathcal{D}} (x+y)\, dA = \int_0^1 \int_0^1 (7u+4v) \cdot 13\, du\, dv$$

$$= 13 \int_0^1 \left(\frac{7}{2}u^2 + 4uv\right)\Big|_0^1 dv$$

$$= 13 \int_0^1 \left(\frac{7}{2} + 4v\right) dv$$

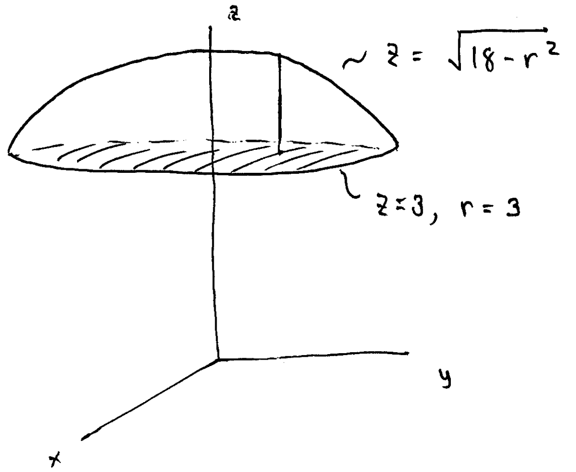
$$= 13\left(\frac{7}{2}v + 2v^2\right)\Big|_0^1$$

$$= 13\left(\frac{7}{2} + 2\right)$$

$$= \frac{143}{2}$$

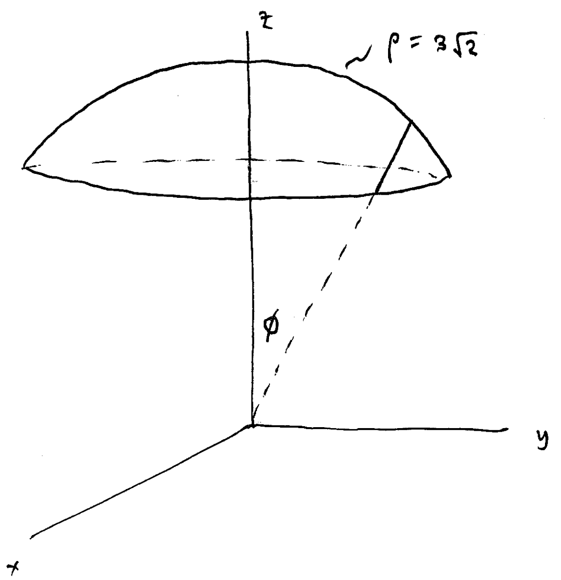
7. (20 points) An object occupying the region defined by the inequalities $x^2 + y^2 + z^2 \leqslant 18$ and $z \geqslant 3$ has mass density $\delta(x, y, z) = 5/z$ kg per cubic unit. Set up (**but do not evaluate**) integrals that give the mass of the object:

(a) using cylindrical coordinates

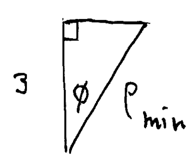


$$M = \int_0^{2\pi} \int_0^{3} \int_3^{\sqrt{18-r^2}} \frac{5}{z} \cdot r \, dz \, dr \, d\theta$$

(b) using spherical coordinates



$$M = \int_0^{2\pi} \int_0^{\pi/4} \int_{3\sec\phi}^{3\sqrt{2}} \frac{5}{\rho\cos\phi} \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



$$\rho_{min} = \frac{3}{\cos\phi} = 3\sec\phi$$

$$\cos\phi_{max} = \frac{3}{\sqrt{18}} = \frac{1}{\sqrt{2}} \Rightarrow \phi_{max} = \pi/4$$