MAT 261—Exam #2—10/16/14

Name: Solutions

Calculators are not permitted. Show all of your work using correct mathematical notation.

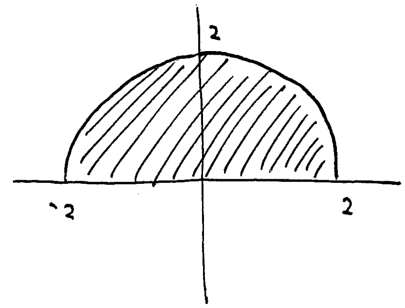
1. (10 points) Find and sketch the domain of the function $f(x, y) = \sqrt{y} + \sqrt{4 - x^2 - y^2}$.

We need $y \geq 0$ and

$4 - x^2 - y^2 \geq 0$

$\Leftrightarrow \quad x^2 + y^2 \leq 4$

So the domain is a half-disk of radius 2



2. (15 points) Let $f(x, y, z) = \dfrac{x^4 \ln z}{y^5} + e^{xy+yz^3} \tan(z^7)$. Calculate f_x, f_y, and f_z.

$$f_x = \frac{4x^3 \ln z}{y^5} + y\, e^{xy+yz^3} \tan(z^7)$$

$$f_y = \frac{-5x^4 \ln z}{y^6} + (x + z^3)\, e^{xy+yz^3} \tan(z^7)$$

$$f_z = \frac{x^4}{y^5 z} + e^{xy+yz^3} \cdot 7z^6 \sec^2(z^7)$$

$$+ \; 3yz^2 e^{xy+yz^3} \tan(z^7)$$

3. (10 points) Let $f(x,y) = \dfrac{x^2y}{(x+y)^3}$. Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Along the x-axis, we have $f(x,0) = 0$ for all $x \neq 0$,

whereas along the line $y = x$ we have

$$f(x,x) = \frac{x^3}{(2x)^3} = \frac{1}{8} \quad \text{for all } x \neq 0 .$$

It follows from the two-path test that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

4. (15 points) Consider the function $f(x,y,z) = \frac{1}{4}x^2y^3z^5$.

(a) Find the directional derivative of f at the point $(1,2,1)$ in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

$$\vec{\nabla} f = \frac{1}{2} x y^3 z^5 \, \hat{\imath} + \frac{3}{4} x^2 y^2 z^5 \, \hat{\jmath} + \frac{5}{4} x^2 y^3 z^4 \, \hat{k}$$

$$\Rightarrow \quad \vec{\nabla} f_{(1,2,1)} = 4\hat{\imath} + 3\hat{\jmath} + 10\hat{k}$$

$$\vec{u} = e_{\vec{v}} = \frac{1}{\sqrt{6}} (2\hat{\imath} + \hat{\jmath} - \hat{k})$$

$$D_{\vec{u}} f_{(1,2,1)} = \frac{1}{\sqrt{6}} (8 + 3 - 10) = \frac{1}{\sqrt{6}}$$

(b) Find the maximum value of the directional derivative of f at the point $(1,2,1)$.

The maximum value is

$$\| \vec{\nabla} f_{(1,2,1)} \| = \sqrt{125} = 5\sqrt{5}$$

(c) Find the equation of the tangent plane to the level surface $f(x,y,z) = 2$ at the point $(1,2,1)$.

Since $\vec{\nabla} f_{(1,2,1)}$ is a normal vector, we get

$$4(x-1) + 3(y-2) + 10(z-1) = 0$$

$$\Leftrightarrow \quad 4x + 3y + 10z = 20$$

5. (10 points) Find the linearization of the function $f(x, y) = x^2 \cos y$ at the point $(3, \pi/3)$.

$$f(3, \pi/3) = 9 \cos \pi/3 = 9/2$$

$$f_x = 2x \cos y \Rightarrow f_x(3, \pi/3) = 6 \cos \pi/3 = 3$$

$$f_y = -x^2 \sin y \Rightarrow f_y(3, \pi/3) = -9 \sin \pi/3 = -\frac{9}{2}\sqrt{3}$$

$$L(x, y) = \frac{9}{2} + 3(x-3) - \frac{9}{2}\sqrt{3}\left(y - \frac{\pi}{3}\right)$$

6. (15 points) Let $w = \dfrac{4}{2x+3y}$, where $x = r\cos\theta$ and $y = r\sin\theta$. Calculate $\partial w/\partial\theta$ at the point $(r, \theta) = (2, 3\pi/4)$.

$$\begin{tikzcd} & w \arrow[dl, no head] \arrow[dr, no head] & \\ x \arrow[dr, no head] & & y \arrow[dl, no head] \\ & r, \theta & \end{tikzcd}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{-4}{(2x+3y)^2} \cdot 2 \cdot (-r\sin\theta) + \frac{-4}{(2x+3y)^2} \cdot 3 \cdot r\cos\theta$$

$$= \frac{4r}{(2x+3y)^2}\left(2\sin\theta - 3\cos\theta\right)$$

$(r, \theta) = (2, 3\pi/4)$

$\Rightarrow$ $x = -\sqrt{2}$,

$y = \sqrt{2}$

Thus

$$\left.\frac{\partial w}{\partial \theta}\right|_{(2, 3\pi/4)} = \frac{8}{(\sqrt{2})^2}\left(\sqrt{2} + \frac{3}{2}\sqrt{2}\right)$$

$$= 4 \cdot \frac{5}{2}\sqrt{2}$$

$$= 10\sqrt{2}$$

7. (13 points) Find the coordinates of all local maxima, local minima, and saddle points of the function $f(x,y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$.

Critical pts : $f_x = 4x + 3y - 5 = 0$

$f_y = 3x + 8y + 2 = 0$

$\Rightarrow \begin{cases} 12x + 9y = 15 \\ 12x + 32y = -8 \end{cases} \Rightarrow -23y = 23$

$\Rightarrow y = -1$

$\Rightarrow x = 2$

So $(2,-1)$ is the only critical point

$f_{xx} = 4 > 0$

$f_{yy} = 8$

$f_{xy} = 3$

$D = 4 \cdot 8 - 3^2 = 23 > 0$

So f has a local minimum at $(2,-1)$

8. (12 points) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = 3x - y$ on the circle $x^2 + y^2 = 40$.

Let $g(x,y) = x^2 + y^2$

$\vec{\nabla} f = \langle 3, -1 \rangle$ and $\vec{\nabla} g = \langle 2x, 2y \rangle$

So $\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow \begin{cases} 3 = 2x\lambda \\ -1 = 2y\lambda \end{cases} \Rightarrow \frac{x}{y} = -3$

$\Rightarrow x = -3y$

The constraint gives

$(-3y)^2 + y^2 = 40 \Rightarrow 10y^2 = 40$

$\Rightarrow y = \pm 2, \ x = \mp 6$

$f(6,-2) = 20$ max value

$f(-6,2) = -20$ min value