MAT 261—Exam #1—9/18/14

Name: Solutions

Calculators are not permitted. Show all work using correct mathematical notation.

1. (10 points) Find parametric equations for the line containing the points $(1, 0, 2)$ and $(3, 1, 5)$.

Let $P = (1, 0, 2)$ and $Q = (3, 1, 5)$.

Then $\vec{PQ} = \langle 2, 1, 3 \rangle$ is a direction vector for the line, so we get

$$\langle x, y, z \rangle = \langle 1, 0, 2 \rangle + \langle 2, 1, 3 \rangle t$$

$$\Rightarrow \quad x = 1 + 2t, \quad y = t, \quad z = 2 + 3t$$

2. (15 points) A particle moves in space with trajectory

$$x(t) = \ln(t^2 + 1), \quad y(t) = \sin(\pi t), \quad z(t) = \frac{3}{t^2}.$$

Find the unit tangent vector (that is, the unit vector tangent to the particle's path) at $t = 1$.

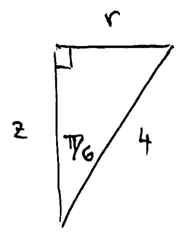
$$x'(t) = \frac{2t}{t^2+1} \qquad y'(t) = \pi \cos(\pi t) \qquad z'(t) = \frac{-6}{t^3}$$

$$\Rightarrow x'(1) = 1 \qquad \Rightarrow y'(1) = -\pi \qquad \Rightarrow z'(1) = -6$$

The velocity vector $\vec{v}(1) = \langle 1, -\pi, -6 \rangle$ is tangent to the path at $t = 1$, so the unit tangent vector is

$$\frac{\vec{v}(1)}{\|\vec{v}(1)\|} = \frac{\langle 1, -\pi, -6 \rangle}{\sqrt{37 + \pi^2}}$$

3. (10 points) Convert the spherical coordinates $(\rho, \theta, \phi) = (4, 3\pi/4, \pi/6)$ into rectangular coordinates (x, y, z).



$$z = 4 \cos \pi/6 = 2\sqrt{3}$$

$$r = 4 \sin \pi/6 = 2$$

$$\Rightarrow \quad x = 2 \cos 3\pi/4 = -\sqrt{2}$$

$$y = 2 \sin 3\pi/4 = \sqrt{2}$$

Thus $(x, y, z) = (-\sqrt{2}, \sqrt{2}, 2\sqrt{3})$

4. (15 points) Find the length of the curve $c(t) = (2t^{3/2} + 1, 5t + 3)$ on the interval $0 \leqslant t \leqslant 1$.

$$L = \int_0^1 \sqrt{(3t^{1/2})^2 + 5^2} \, dt$$

$$= \int_0^1 \sqrt{9t + 25}$$

Let $u = 9t + 25$

$du = 9 \, dt$

$$= \frac{1}{9} \int_{25}^{34} \sqrt{u} \, du$$

$$= \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_{25}^{34}$$

$$= \frac{2}{27} \left(34^{3/2} - 125 \right)$$

5. (10 points) Find an equation for the plane passing through the origin and containing the vectors $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 1 \\ 2 & 1 & 5 \end{vmatrix} = 4\hat{i} - 13\hat{j} + \hat{k}$$

is a normal vector to the plane.

Since the plane passes through the origin, it has equation

$$4x - 13y + z = 0.$$

6. (15 points) Consider the points $P(0,1)$, $Q(1,3)$, and $R(1,5)$ in $\mathbb{R}^2$.

(a) Find the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\overrightarrow{PQ} = \langle 1, 2 \rangle \qquad \overrightarrow{PR} = \langle 1, 4 \rangle$$

$$\cos\theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\|\overrightarrow{PQ}\| \, \|\overrightarrow{PR}\|}$$

$$= \frac{9}{\sqrt{5} \cdot \sqrt{17}} \implies \theta = \cos^{-1}\left(\frac{9}{\sqrt{85}}\right)$$

(b) Find the area of the triangle with vertices P, Q, and R.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 1 & 4 & 0 \end{vmatrix} = 2\hat{k}$$

(embedding the vectors into $\mathbb{R}^3$)

Thus $\text{Area}(\triangle PQR) = \frac{1}{2} \|\overrightarrow{PQ} \times \overrightarrow{PR}\| = 1$

7. (15 points) Consider the plane $x + 2y + 3z = 4$.

(a) Find a unit vector perpendicular to the plane.

$\vec{n} = \langle 1, 2, 3 \rangle$ is a normal vector,

so $\vec{e}_{\vec{n}} = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle$ is a unit normal vector.

(b) Find the point at which the line $\mathbf{r}(t) = \langle 2 + 3t, 1 + t, 4 - t \rangle$ intersects the plane.

Substitution gives

$$2 + 3t + 2(1 + t) + 3(4 - t) = 4.$$

$$\Rightarrow \quad 2t = -12$$

$$\Rightarrow \quad t = -6$$

So the intersection point is

$$(-16, -5, 10).$$

8. (10 points) Find a formula for the speed of a particle moving along the helix

$$\mathbf{r}(t) = \langle kt, A\cos\omega t, A\sin\omega t \rangle.$$

Your formula should involve the constants k, A, and ω and should be given in simplest possible form.

$$\vec{r}\,'(t) = \langle k, -A\omega \sin \omega t, A\omega \cos \omega t \rangle$$

$$\Rightarrow \quad \|\vec{r}\,'(t)\| = \sqrt{k^2 + (A\omega)^2(\sin^2 \omega t + \cos^2 \omega t)}$$

$$= \sqrt{k^2 + (A\omega)^2}$$