MAT 162—Exam #3—4/24/15

Name: Solutions

Show all work using correct mathematical notation. Calculators are not permitted.

1. (10 points) Find the limit of the sequence

$$a_n = \frac{\ln(7n^3 + 8n + 1)}{\ln(5n^2 + 4)}.$$

Show your work using correct limit notation, and simplify your answer.

By L'Hopital's Rule,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{21n^2 + 8}{7n^3 + 8n + 1} \cdot \frac{5n^2 + 4}{10n}$$

$$= \frac{105}{70} \quad \text{by leading coefficients}$$

$$= \frac{3}{2}$$

2. (10 points) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{5}{2^{3n+1}}.$$

Give your answer in simplest possible form.

Geometric with $c = \frac{5}{16}$ and $r = \frac{1}{8}$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2^{3n+1}} = \frac{5/16}{1 - 1/8}$

$$= \frac{5}{16} \cdot \frac{8}{7}$$

$$= \frac{5}{14}$$

3. (10 points) Consider the sequence $a_n = 2 + e^{-n}$. Evaluate

(a) $\lim_{n\to\infty} a_n$ $= 2$ since $e^{-n} = \frac{1}{e^n} \to 0$ as $n \to \infty$

(b) $\sum_{n=1}^{\infty} a_n$ $= \infty$ by the Divergence Test since $\lim_{n\to\infty} a_n \neq 0$.

4. (20 points) Consider the series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$.

(a) Determine whether the series converges absolutely, converges conditionally, or diverges. Justify your answer using appropriate tests.

The sequence $a_n = \frac{1}{\sqrt{n}}$ is positive, decreasing, and approaches 0, so the series converges by the Alternating Series Test.

However, $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series $(p = \frac{1}{2} \leq 1)$, so the series S converges conditionally.

(b) Write out the fifth partial sum, S_5.

$$S_5 = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{2} + \frac{1}{\sqrt{5}}$$

(c) If we use the approximation $S \approx S_5$, what is the maximum possible error in our estimate?

By the AST Estimation Theorem,

$$|S - S_5| < a_6 = \frac{1}{\sqrt{6}}$$

5. (30 points) Decide whether each series converges or diverges, and justify your conclusions using appropriate tests. You must give coherent arguments to receive credit.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^8 n}{\sqrt{n^5+1}}$

Since $\cos^8 n \le 1$ and

$\sqrt{n^5+1} \geq \sqrt{n^5} = n^{5/2}$, we have

$\dfrac{\cos^8 n}{\sqrt{n^5+1}} \le \dfrac{1}{n^{5/2}}$ so the series converges by direct comparison with a p-series $(p = 5/2 > 1)$.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/3}}$

We have $\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^{1/3}} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^{1/3}}$

$u = \ln x$
$du = \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u^{-1/3}\,du$$

$$= \lim_{b\to\infty} \frac{3}{2}\left[(\ln b)^{2/3} - (\ln 2)^{2/3}\right]$$

$$= \infty$$

So the series diverges by the Integral Test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)!\,(n+2)!} \cdot \frac{n!\,(n+1)!}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+2)}$$

$$= 4 > 1$$

So the series diverges by the Ratio Test.

6. (20 points) Find the radius and interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(x+4)^n}{(2n+3)7^n}.$$

Justify your conclusions by citing appropriate tests.

Apply Ratio Test :

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(x+4)^{n+1}}{(2n+5)\cdot 7^{n+1}} \cdot \frac{(2n+3)\cdot 7^n}{(x+4)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x+4}{7} \cdot \frac{2n+3}{2n+5} \right|$$

$$= \frac{|x+4|}{7}$$

Hence the series converges if $|x+4| < 7$ and diverges if $|x+4| > 7$, so $R = 7$.

Endpoints :

$x = 3$: $\displaystyle\sum_{n=1}^{\infty} \frac{7^n}{(2n+3)\,7^n} = \sum_{n=1}^{\infty} \frac{1}{2n+3}$ diverges

by limit comparison with the p-series

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ since $\dfrac{2n+3}{n} \to 2$ as $n \to \infty$

$x = -11$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-7)^n}{(2n+3)\,7^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+3}$

converges by the AST.

Hence $I = [-11, 3)$.