MAT 161—Sample Final Exam

Name: Solutions

Calculators are NOT allowed. Show all work using correct mathematical notation.

1. Calculate each of the following limits.

(a) $\displaystyle\lim_{x\to 0}\frac{e^{3x}-1}{x} \underset{L'H}{=} \lim_{x\to 0}\frac{3e^{3x}}{1} = 3$

"$\frac{0}{0}$"

(b) $\displaystyle\lim_{x\to 1+}\frac{x^2-4}{x-1} = -\infty$

"$\frac{-}{0^+}$"

(c) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x} \underset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0$

"$\frac{\infty}{\infty}$"

(d) $\displaystyle\lim_{x\to 5}\frac{\sqrt{3x+1}-4}{x-5}\cdot\frac{\sqrt{3x+1}+4}{\sqrt{3x+1}+4} = \lim_{x\to 5}\frac{3x+1-16}{(x-5)(\sqrt{3x+1}+4)}$

"$\frac{0}{0}$"

$\displaystyle = \lim_{x\to 5}\frac{3(x-5)}{(x-5)(\sqrt{3x+1}+4)}$

$\displaystyle = \lim_{x\to 5}\frac{3}{\sqrt{3x+1}+4} = \frac{3}{8}$

2. Sketch the graph of a function $y = f(x)$ with the following properties:

(i) f is continuous but not differentiable at $x = -1$

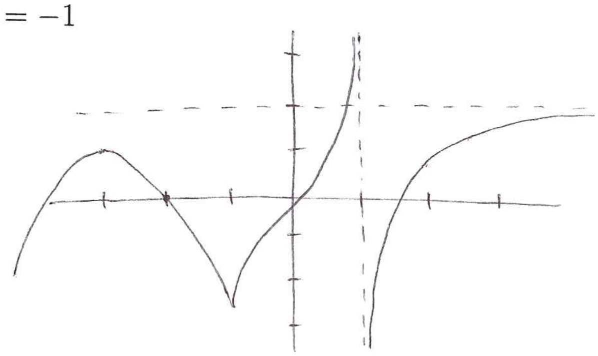
(ii) $f(-2) = 0$

(iii) $f'(-3) = 0$

(iv) $\displaystyle\lim_{x\to 1+} f(x) = -\infty$

(v) $\displaystyle\lim_{x\to\infty} f(x) = 2$

one possibility:



3. Find the derivative of each of the following functions.

(a) $f(x) = x \ln x$

$$f'(x) = x \cdot \frac{1}{x} + \ln x$$
$$= 1 + \ln x$$

(b) $g(x) = \dfrac{\cos(x^3)}{x}$

$$g'(x) = \frac{x\left(-\sin(x^3)\right)\cdot 3x^2 - \cos(x^3)}{x^2}$$
$$= \frac{-3x^3 \sin(x^3) - \cos(x^3)}{x^2}$$

(c) $h(x) = (1 + \sin^4 x)^{2/3}$

$$h'(x) = \frac{2}{3}\left(1+\sin^4 x\right)^{-1/3} \cdot 4\sin^3 x \cdot \cos x$$
$$= \frac{8\sin^3 x \cos x}{3\sqrt[3]{1+\sin^4 x}}$$

(d) $w(x) = x^{\tan x}$

$$\ln w(x) = \tan x \ln x$$
$$\Rightarrow \quad \frac{1}{w(x)} \cdot w'(x) = (\tan x)\cdot \frac{1}{x} + (\ln x)\cdot \sec^2 x$$
$$\Rightarrow \quad w'(x) = x^{\tan x}\left(\frac{\tan x}{x} + (\ln x)\cdot \sec^2 x\right)$$

(e) $F(x) = \displaystyle\int_2^x \sqrt{t^3+1}\,dt$

$$F'(x) = \sqrt{x^3+1}$$ by FTC, Part II

4. Evaluate each of the following integrals.

(a) $\int \left(\frac{3}{x} + \frac{4}{x^2}\right) dx$

$= 3 \ln|x| - \frac{4}{x} + C$

(b) $\int x^4 e^{x^5}\, dx \quad = \frac{1}{5}\int e^u \, du$

$u = x^5 \quad = \frac{1}{5} e^u + C$

$du = 5x^4 \, dx$

$= \frac{1}{5} e^{x^5} + C$

(c) $\int_0^1 (x^2 + \sqrt{x})\, dx$

$= \frac{x^3}{3} + \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{1}{3} + \frac{2}{3} = 1$

(d) $\int_0^{\pi/2} \sin^6 x \cos x \, dx \quad = \int_0^1 u^6 \, du$

$u = \sin x \quad = \frac{u^7}{7}\Big|_0^1 = \frac{1}{7}$

$du = \cos x \, dx$

(e) $\int_0^3 \frac{dx}{9 + x^2} \quad = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\Big|_0^3$

$= \frac{1}{3}\left(\tan^{-1}(1) - \tan^{-1}(0)\right)$

$= \frac{\pi}{12}$

5. Consider the function $f(x) = x^3 - 3x^2 + 1$.

(a) Find the intervals of increase or decrease.

(b) Find the intervals of concavity.

(c) Sketch a graph of the function, clearly labeling the coordinates of all local extrema and inflection points.

$$f'(x) = 3x^2 - 6x$$
$$= 3x(x-2)$$

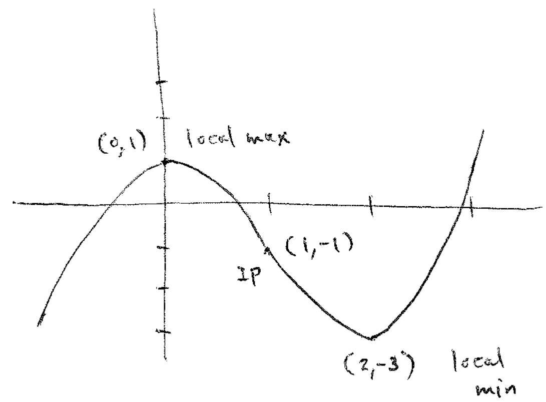
$+ \quad - \quad +$ f'

$\nearrow$ 0 $\searrow$ 2 $\nearrow$ f

$$f''(x) = 6x - 6$$

$- \quad +$ f''

$\frown$ 1 $\smile$ f



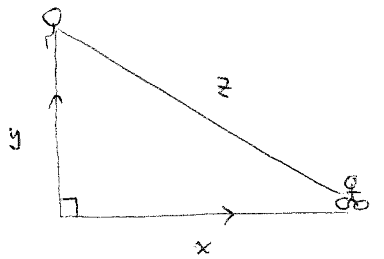
6. State the definition of the derivative in terms of a limit, and use it to calculate the derivative of $f(x) = 3x^2 + 5x$. No credit will be given for shortcut methods.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{3(x+h)^2 + 5(x+h) - (3x^2 + 5x)}{h}$$
$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 + 5x + 5h - 3x^2 - 5x}{h}$$
$$= \lim_{h \to 0} \frac{6xh + 3h^2 + 5h}{h} = \lim_{h \to 0} (6x + 5 + 3h)$$
$$= 6x + 5$$

7. Find the equation of the tangent line to the curve $x^4 + x^2y + y^3 = 11$ at the point $(1, 2)$.

$$4x^3 + x^2 \frac{dy}{dx} + y \cdot 2x + 3y^2 \frac{dy}{dx} = 0$$
$$\Rightarrow (x^2 + 3y^2) \frac{dy}{dx} = -4x^3 - 2xy$$
$$\Rightarrow \frac{dy}{dx} = \frac{-4x^3 - 2xy}{x^2 + 3y^2}$$
$$\Rightarrow \left.\frac{dy}{dx}\right|_{(1,2)} = \frac{-4-4}{1+12} = -\frac{8}{13}$$

8. A balloon is rising at a constant speed of 5 ft/sec. A boy is cycling along a straight road at a speed of 15 ft/sec. When the boy passes under the balloon, it is 45 feet above him. How fast is the distance between the boy and the balloon increasing 3 seconds later?



$$x^2 + y^2 = z^2$$

$$\Rightarrow \quad 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$$

$$\Rightarrow \quad 45 \cdot 15 + 60 \cdot 5 = 75 \frac{dz}{dt}$$

Given: $\frac{dy}{dt} = 5$

$\frac{dx}{dt} = 15$

$$\Rightarrow \quad \frac{dz}{dt} = \frac{45 \cdot 15 + 60 \cdot 5}{75}$$

Want: $\frac{dz}{dt}$ when

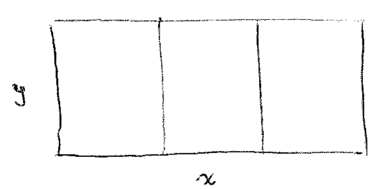
$$= 13$$

$x = 45, \quad y = 60,$

$z = 75$

The distance is increasing at a rate of 13 ft/sec at this instant

9. A farmer with 750 feet of fencing wants to enclose a rectangular area and then divide it into three pens with fencing parallel to one side of the rectangle. Find the largest possible total enclosed area, and justify that your solution is a maximum.



Want to maximize $A = xy$ subject to the constraint $2x + 4y = 750$.

Then $x = 375 - 2y$, so

$$A(y) = (375 - 2y)\, y$$

$$= 375y - 2y^2$$

$$\Rightarrow \quad A'(y) = 375 - 4y = 0$$

$$\Rightarrow \quad y = \frac{375}{4} \quad \Rightarrow \quad x = \frac{375}{2}$$

Since $A''(y) = -4 < 0$, these dimensions give a maximum area of $\frac{(375)^2}{8}$ square feet.

10. The acceleration of a particle (in m/s^2) after t seconds is given by $a(t) = 2t + 4$, and the particle's velocity at $t = 1$ is 7 m/s.

(a) Find the particle's velocity function $v(t)$.

$v(t) = t^2 + 4t + c_1$ and $v(1) = 7$

$\Rightarrow 1 + 4 + c_1 = 7 \Rightarrow c_1 = 2$

So $v(t) = t^2 + 4t + 2$

(b) Find the distance traveled by the particle between $t = 0$ and $t = 10$.

$s(10) - s(0) = \int_0^{10} (t^2 + 4t + 2)\, dt$

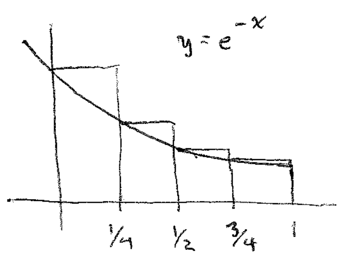
net displacement = distance traveled since $v(t) \geq 0$ on $[0, 10]$

$= \frac{t^3}{3} + 2t^2 + 2t \Big|_0^{10}$

$= \frac{1000}{3} + 200 + 20 = \frac{1660}{3}$ meters

11. Consider the area under the curve $f(x) = e^{-x}$ on the interval $[0, 1]$.

(a) Compute the estimate L_4, and show your rectangles on a sketch of the region.



$L_4 = e^0 \cdot \frac{1}{4} + e^{-1/4} \cdot \frac{1}{4} + e^{-1/2} \cdot \frac{1}{4} + e^{-3/4} \cdot \frac{1}{4}$

$= \frac{1}{4}\left(1 + e^{-1/4} + e^{-1/2} + e^{-3/4}\right)$

(b) Which of the following gives the exact area under the curve?

$\Delta x = 1/N$

$x_j = j/N$

(i) $\lim_{N\to\infty} \sum_{j=1}^{N} \frac{e^{-j/N}}{N}$ (ii) $\lim_{x\to 1} \sum_{j=0}^{x} je^{-j}$ (iii) $\lim_{N\to 0} \sum_{j=1}^{N} \frac{e^{-j/N}}{N}$ (iv) $\lim_{x\to\infty} \sum_{j=0}^{x} je^{-j}$

12. Given that $\int_1^7 f(x)\, dx = 9$ and $\int_1^4 f(x)\, dx = 13$, evaluate $\int_4^7 (2f(x) + 5)\, dx$.

$\int_4^7 (2f(x) + 5)\, dx = 2\int_4^7 f(x)\, dx + \int_4^7 5\, dx$

$= 2\left(\int_1^7 f(x)\, dx - \int_1^4 f(x)\, dx\right) + 5x \Big|_4^7$

$= 2(9 - 13) + 5(7 - 4)$

$= 7$