MAT 161-03—Exam #3—11/16/10

Name: Solutions

Calculators are NOT allowed. Show all work using correct mathematical notation.

1. (20 points) Consider the function $f(x) = x^5 - 5x^4$.

   (a) Determine the intervals on which f is increasing/decreasing.

   (b) Determine the intervals on which f is concave up/concave down.

   (c) Sketch a graph of the function, clearly labeling the coordinates of all intercepts, local extrema, and inflection points.

a) $f'(x) = 5x^4 - 20x^3$

$= 5x^3(x-4)$

$+$		$-$		$+$		f'
$\nearrow$	0	$\searrow$	4	$\nearrow$		f

f is increasing on $(-\infty, 0) \cup (4, \infty)$

decreasing on $(0, 4)$

b) $f''(x) = 20x^3 - 60x^2$

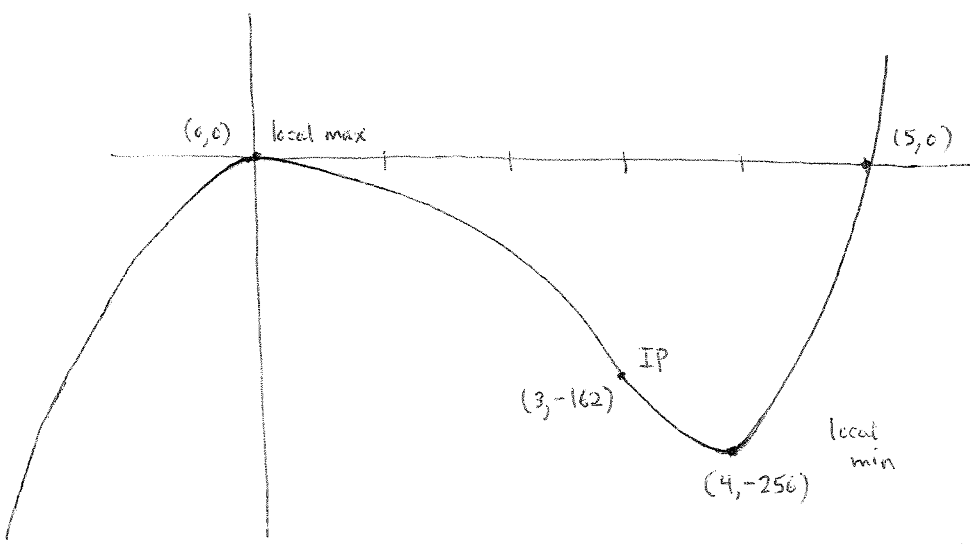
$= 20x^2(x-3)$

$-$		$-$		$+$		f''
$\frown$	0	$\frown$	3	$\smile$		f

f is concave up on $(3, \infty)$

concave down on $(-\infty, 0) \cup (0, 3)$

c)



2. (12 points) Evaluate each of the following limits. Show all work using correct notation!

(a) $\lim\limits_{x\to\infty} \dfrac{\ln x}{x}$ $\underset{L'H}{=}$ $\lim\limits_{x\to\infty} \dfrac{1/x}{1}$

" $\dfrac{\infty}{\infty}$ "

$= 0$

(b) $\lim\limits_{x\to 0} \dfrac{1-\cos 2x}{1-\cos 3x}$ $\underset{L'H}{=}$ $\lim\limits_{x\to 0} \dfrac{2\sin 2x}{3\sin 3x}$

" $\dfrac{0}{0}$ " $\underset{L'H}{=}$ $\lim\limits_{x\to 0} \dfrac{4\cos 2x}{9\cos 3x}$

$= \dfrac{4}{9}$

3. (8 points) Find the equations of all horizontal and vertical asymptotes of the function $f(x) = \dfrac{45-5x^2}{3x^2-12}$.

HORIZONTAL: $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \dfrac{\frac{45}{x^2}-5}{3-\frac{12}{x^2}} = -\dfrac{5}{3}$

So $y = -5/3$ is the horizontal asymptote

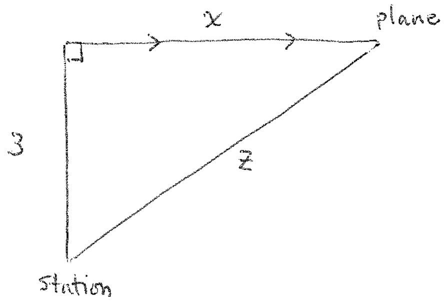
VERTICAL: $3x^2 - 12 = 0$

$\Leftrightarrow x^2 = 4$

$\Leftrightarrow x = \pm 2$ vertical asymptotes since $45 - 5(\pm 2)^2 \neq 0$

4. (20 points) A plane is traveling at a constant altitude of 3 miles and a constant speed of 240 mph. At noon, the plane passes directly over a radar station.

(a) How fast is the distance between the plane and the radar station changing at 12:01 pm?



$$3^2 + x^2 = z^2$$

$$\Rightarrow \quad 2x \frac{dx}{dt} = 2z \frac{dz}{dt}$$

$$\Rightarrow \quad \frac{dz}{dt} = \frac{x}{z} \cdot \frac{dx}{dt}$$

$$= \frac{4}{5} \cdot 240$$

$$= 192$$

Given : $\frac{dx}{dt} = 240 \ \text{mi/hr}$

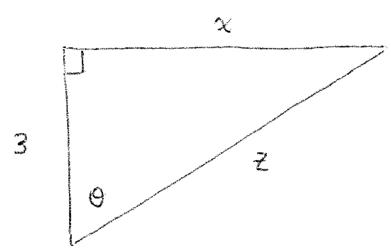
At 12:01 pm,

$$x = 240 \cdot \frac{1}{60} = 4$$

$$\Rightarrow \quad z = \sqrt{9 + 16} = 5$$

At 12:01 pm, the distance is increasing at a rate of 192 mph.

(b) How fast is the angle between the vertical and the station's line of sight to the plane changing at 12:01 pm? Give your answer in radians per minute.



$$\tan\theta = \frac{x}{3}$$

$$\Rightarrow \quad \sec^2\theta \frac{d\theta}{dt} = \frac{1}{3} \frac{dx}{dt}$$

$$\Rightarrow \quad \frac{d\theta}{dt} = \frac{1}{3} \cos^2\theta \frac{dx}{dt}$$

$$= \frac{1}{3} \left(\frac{3}{5}\right)^2 \cdot 4$$

$$= 12/25$$

Note :

$240 \ \text{mi/hr} = 4 \ \text{mi/min}$

At 12:01 pm, the angle is increasing at a rate of $12/25$ rad/min.

5. (8 points) Find the absolute maximum and minimum values of $f(x) = x - 2\sin x$ on the interval $[0, \pi]$.

$$f'(x) = 1 - 2\cos x = 0$$

$$\Rightarrow \cos x = \frac{1}{2}$$

$$\Rightarrow x = \pi/3$$

only critical point in $[0, \pi]$

Test critical point & endpoints:

$f(0) = 0$

$f(\pi/3) = \pi/3 - \sqrt{3}$ MIN

$f(\pi) = \pi$ MAX

6. (12 points) Evaluate each of the following indefinite integrals.

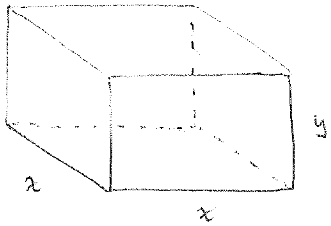
(a) $\displaystyle\int (x^5 + e^{5x} + 3)\, dx$

$$= \frac{1}{6}x^6 + \frac{1}{5}e^{5x} + 3x + C$$

(b) $\displaystyle\int \left(\frac{3}{x} - \frac{6}{x^3} + 4\cos 7x\right) dx$

$$= 3\ln|x| + \frac{3}{x^2} + \frac{4}{7}\sin 7x + C$$

7. (20 points) You are asked to design a box of volume 6 cubic feet with square base and no top. The material for the bottom costs \$3 per square foot, and the material for the sides costs \$2 per square foot. Find the dimensions of the box that minimize the total cost, and justify that your solution gives a minimum.



Want to minimize

$$C = 3 \cdot x^2 + 2 \cdot 4xy$$
$$= 3x^2 + 8xy$$

subject to the constraint

$$x^2 y = 6 \quad \Rightarrow \quad y = 6/x^2 .$$

Thus $C(x) = 3x^2 + 8x \cdot (6/x^2)$

$$= 3x^2 + \frac{48}{x}$$

$$C'(x) = 6x - \frac{48}{x^2} = 0$$

$$\Rightarrow \quad x^3 = 8$$

$$\Rightarrow \quad x = 2 \quad \text{critical pt}$$

$$C''(x) = 6 + \frac{96}{x^3} > 0 \quad \text{when } x > 0 \quad \cup$$

So the critical pt gives a local minimum.

Since there are no other critical points, this must in fact give the absolute minimum.

Thus we should take $x = 2$ and $y = 6/2^2 = 1.5$.

Hence the base of our box should be

2 ft by 2 ft, and the height should be 1.5 ft.