MAT 161—West Chester University—Fall 2010
Notes on Rogawski's Calculus: Early Transcendentals
Scott Parsell

§2.1—Limits, Rates of Change, and Tangent Lines

Example 1. The distance in feet that an object falls in t seconds under the influence of gravity is given by $s(t) = 16t^2$. If a pumpkin is dropped from the top of a tall building, calculate its average velocity

(a) between $t = 3$ and $t = 4$

(b) between $t = 3$ and $t = 3.5$

(c) between $t = 3$ and $t = 3.1$

(d) between $t = 3$ and $t = 3.01$

(e) between $t = 3$ and $t = 3.001$

What appears to be the instantaneous velocity of the pumpkin at $t = 3$?

In general, if an object's position is given by $s(t)$, then the **average velocity** over the interval $[t_0, t_1]$ is

$$\frac{\Delta s}{\Delta t} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}.$$

By allowing the point t_1 to move closer and closer to t_0, we are able to estimate the **instantaneous velocity** at t_0. This process of computing what happens as t_1 gets very close to (but never equal to) t_0 is an example of taking a **limit**.

Even more generally, the **average rate of change** of the function $y = f(x)$ over the interval $[x_0, x_1]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Geometrically, it is the slope of the secant line connecting the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

By allowing the point x_1 to move closer and closer to x_0, we obtain the **tangent line** to the graph of $y = f(x)$ at the point $x = x_0$. The slope of the tangent line at x_0 is the **instantaneous rate of change** of y with respect to x at the point x_0.

Example 2. Find the slope of the secant line on the graph of $f(x) = e^x$ for each of the following intervals.

(a) $[0, 0.5]$

(b) $[0, 0.1]$

(c) $[0, 0.01]$

(d) $[0, 0.001]$

What appears to be the slope of the tangent line to the graph at $x = 0$? What is the equation of this tangent line?

§2.2—Limits: A Numerical and Graphical Approach

As we saw in the previous section, in order to make sense of instantaneous rates of change, we need to understand the concept of limits.

Definition. If we can make $f(x)$ as close as we like to L by taking x sufficiently close to c, then we say that the **limit** of $f(x)$ as x approaches c is equal to L, and we write

$$\lim_{x\to c} f(x) = L.$$

In other words, this statement means that the quantity $|f(x) - L|$ becomes arbitrarily small (but not necessarily zero) whenever x is sufficiently close to (but not equal to) c.

Note: In the previous section, we used numerical data to analyze

$$\lim_{t\to 3} \frac{16t^2 - 144}{t-3} \qquad \text{and} \qquad \lim_{x\to 0} \frac{e^x - 1}{x}.$$

Important points:

1. We must consider values on both sides of c.
2. The limit may or may not exist.
3. The value of f at $x = c$ is irrelevant.

Example 1. Use numerical and graphical data to guess the values of the following limits.

(a) $\lim_{x\to 1} \dfrac{x^2 - 1}{x - 1}$

(b) $\lim_{x\to 0} \dfrac{\sin x}{x}$

One-sided limits

$\lim_{x \to c^+} f(x) = L$ means $f(x)$ approaches L as x approaches c from the right
$\lim_{x \to c^-} f(x) = L$ means $f(x)$ approaches L as x approaches c from the left

We have $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^+} f(x) = L$ and $\lim_{x \to c^-} f(x) = L$.

Example 2. Compute each of the following limits for the function graphed below.

(a) $\lim_{x \to 2^+} f(x)$ (b) $\lim_{x \to 2^-} f(x)$ (c) $\lim_{x \to 2} f(x)$

(d) $\lim_{x \to 5^+} f(x)$ (e) $\lim_{x \to 5^-} f(x)$ (f) $\lim_{x \to 5} f(x)$

Infinite Limits

We write $\lim_{x \to c} f(x) = \infty$ if the values of $f(x)$ become arbitrarily large and positive as x approaches c. Similarly, we write $\lim_{x \to c} f(x) = -\infty$ if the values of $f(x)$ become arbitrarily large and negative as x approaches c. Similar definitions apply to one-sided infinite limits.

Notice that if $\lim_{x \to c^+} f(x) = \pm\infty$ or $\lim_{x \to c^-} f(x) = \pm\infty$ then the line $x = c$ is a vertical asymptote for the graph of $y = f(x)$.

Example 3. Evaluate each of the following limits.

(a) $\lim_{x \to 0^-} \frac{1}{x}$ (b) $\lim_{x \to 2} \frac{1}{(x-2)^4}$

(c) $\lim_{x \to 5^+} \frac{x+3}{x-5}$ (d) $\lim_{x \to 0^+} \ln x$

§2.3—Basic Limit Laws

While numerical data and graphs often provide useful intuition about limits, relying exclusively on this type of information can give misleading results if a function has subtle behavior that is not captured by our data. Therefore, we need to develop explicit methods for computing limits. The following example illustrates some of the basic principles.

Example 1. Evaluate each of the following limits.

(a) $\lim_{x\to 4} 7$

(c) $\lim_{x\to 4} (3x+7)$

(b) $\lim_{x\to 4} x$

From the reasoning in parts (a) and (b) we see that for any constants k and c

$$\lim_{x\to c} k = k \qquad \text{and} \qquad \lim_{x\to c} x = c.$$

We can also generalize part (c). Assuming $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, we have

(1) Sums and Differences: $\lim_{x\to c} (f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$

(2) Constant Multiples: For any constant k, we have $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$

(3) Products: $\lim_{x\to c} f(x)g(x) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$

(4) Quotients: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ provided that $\lim_{x\to c} g(x) \neq 0$.

Example 2. Use the above limit laws to evaluate $\lim_{x\to 2} \dfrac{x^4+3x^2}{4x+5}$.

§2.4—Limits and Continuity

The reasoning from Example 2 of §2.3 shows that the limit of any polynomial or rational function can be found by direct substitution, provided the limit of the denominator is not zero. This property is known as *continuity* and is shared by many familiar types of functions.

Definition. We say that f is **continuous** at $x = c$ if $\lim_{x \to c} f(x) = f(c)$.

This implicitly requires checking three things:

(i) $f(c)$ exists (ii) $\lim_{x \to c} f(x)$ exists (iii) the numbers in (i) and (ii) are equal.

If c is an endpoint of the domain, we use the appropriate one-sided limit instead. A point where f is not continuous is called a **discontinuity**.

Example 1. At what points does the function graphed below fail to be continuous?

Notice that there are various ways in which a function can fail to be continuous–for example, a hole in the graph, a finite jump, or a vertical asymptote.

Example 2. Sketch the graphs of the following functions near $x = 2$.

(a) $f(x) = \dfrac{x^2 - 4}{x - 2}$

(c) $f(x) = \dfrac{x^2 + 4}{x - 2}$

(b) $f(x) = \dfrac{x^2 - 4}{|x - 2|}$

Useful facts:

1. Sums, differences, products, quotients, powers, roots, and compositions of continuous functions are continuous at all points of their domains.

2. Polynomials, rational functions, root functions, trigonometric functions, exponentials, and logarithms are continuous at all points of their domains.

Example 3. For what values of x is the function $f(x) = \dfrac{2+\sin(x^2)}{\sqrt{x^2-4}-1}$ continuous?

Example 4. For what values of x is the function $f(x) = \begin{cases} x+1 & \text{if } x \leq 3 \\ x^2-1 & \text{if } x > 3 \end{cases}$ continuous?

Note that the definition of $f(x)$ in Example 4 automatically ensures that the limit as x approaches 3 from the left is equal to $f(3)$, so we say that f is **left-continuous** at $x = 3$.

Example 5. For what values of a is $f(x) = \begin{cases} ax+1 & \text{if } x \leq 3 \\ ax^2-1 & \text{if } x > 3 \end{cases}$ continuous at $x = 3$?

§2.5—Evaluating Limits Algebraically

We saw in Example 2 of §2.4 that the behavior of a function near a point where both the numerator and denominator approach zero can sometimes be analyzed by cancelling a common factor. These "0/0" limits occur frequently when dealing with instantaneous rates of change, so we illustrate here some of the algebraic manipulations that can be useful.

Example 1. Evaluate $\lim_{x \to 3} \dfrac{x^2 - 9}{x^2 - 2x - 3}$.

Example 2. Evaluate $\lim_{x \to 25} \dfrac{25 - x}{5 - \sqrt{x}}$.

Example 3. Evaluate $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1}$.

Example 4. Evaluate $\lim_{x \to 4} \dfrac{\sqrt{2x+1} - 3}{x - 4}$.

Example 5. Evaluate $\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$, where $f(x) = \dfrac{1}{x}$.

Note: This is the instantaneous rate of change of f at $x = 2$.

§2.6—Trigonometric Limits

Example 1. Calculate $\lim_{x\to 0} x^2 \sin(1/x)$.

The reasoning used in Example 1 is a special case of the following theorem.

The Squeeze Theorem. Suppose that $l(x) \le f(x) \le u(x)$ for all $x \neq c$ in some open interval containing c. If $\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L$ then $\lim_{x\to c} f(x) = L$.

An important application of the Squeeze Theorem is the following result, which we predicted numerically in Example 1(b) of §2.2.

Theorem. If θ is measured in radians, then $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

Example 2. Evaluate the following limits.

(a) $\lim_{x\to 0} \frac{\sin 3x}{x}$

(b) $\lim_{x\to 0} \frac{\tan x}{2x}$

Example 3. Evaluate $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}$

Review of the Unit Circle. If your trigonometry is rusty, now would be a good time to check out §1.4 in the text. In particular, recall that the x and y coordinates of a point on the unit circle at an angle θ (measured counter-clockwise from the positive x-axis) are given by $x = \cos\theta$ and $y = \sin\theta$. Hence the Pythagorean Theorem immediately gives the identity

$$\cos^2\theta + \sin^2\theta = 1.$$

Moreover, by generalizing to a circle of radius r, we obtain the familiar right-triangle relationships given by SOHCAHTOA. You should also remember the values of $\sin\theta$ and $\cos\theta$ at the special angles in the first quadrant:

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

It's then easy to move to other quadrants using a reference angle, remembering that sine is positive in Quadrants I and II and cosine is positive in Quadrants I and IV. Everything about the other four trig functions follows from what we know about sine and cosine via

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}.$$

In particular, it's easy to show that

$$1 + \tan^2\theta = \sec^2\theta \quad \text{and} \quad \cot^2\theta + 1 = \csc^2\theta.$$

§2.7—The Intermediate Value Theorem

An important property of continuous functions is that they do not "skip over" any y-values. The precise statement is as follows:

The Intermediate Value Theorem. Suppose that f is continuous on $[a, b]$ and $f(a) \neq f(b)$. Then for every value M between $f(a)$ and $f(b)$, there exists at least one value c in the interval (a, b) for which $f(c) = M$.

Example 1. Show that the function $f(x) = x^4 + x^2 + 1$ takes on the value 10 for some x in the interval $(1, 2)$.

Root-finding. An important corollary of the IVT is that if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x) = 0$ has a solution in (a, b). By applying this repeatedly, one can find roots of equations to arbitrary accuracy. The algorithm, known as the Bisection Method, is illustrated in the following example.

Example 2. Find an interval of length $1/4$ in which the equation $x^3 + x + 1 = 0$ has a solution.

§3.1—Definition of the Derivative

The slope of the **secant line** connecting the points $P(a, f(a))$ and $Q(a+h, f(a+h))$ on the graph of f is

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}.$$

This is the **average rate of change** of f over the interval $[a, a+h]$.

The slope of the **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists. This is the **instantaneous rate of change** of f at $x = a$ and is also called the **derivative** of f at $x = a$. By setting $x = a + h$, we can alternatively write

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

which is sometimes easier to work with.

Example 1. Find the derivative of the function $f(x) = 16x^2$ at $x = 3$.

Example 2. Find the equation of the tangent line to $f(x) = \sqrt{x+1}$ at the point $(3, 2)$.

Example 3. Find the equation of the tangent line to $f(x) = 1/x^2$ at the point $(-1, 1)$.

§3.2—The Derivative as a Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The process of calculating a derivative is called **differentiation**.

We sometimes write $\frac{dy}{dx}$ or $\frac{d}{dx}[f(x)]$ instead of $f'(x)$.

Example 1. Use the above definition to find the derivative of the following functions.

(a) $f(x) = x^3$

(b) $f(x) = \frac{1}{\sqrt{x}}$

Some Basic Rules:

1. Derivative of a Constant Function: $\frac{d}{dx}(c) = 0$
2. Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$ when n is a constant.
3. Derivative of the Natural Exponential Function: $\frac{d}{dx}(e^x) = e^x$
4. Constant Multiples: $\frac{d}{dx}[cf(x)] = cf'(x)$
5. Sums and Differences: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Example 2. Find the derivative of each of the following functions.

(a) $f(x) = 3x^5 - 4x^3 + \frac{5}{x^2} + 6$

(b) $f(x) = 4e^x + 10\sqrt[3]{x} + \frac{e}{x^3}$

A function $f(x)$ is **differentiable** at $x = c$ if $f'(c)$ exists. There are several ways a function can fail to be differentiable:

1. Corner
2. Cusp
3. Vertical tangent
4. Discontinuity

Theorem. Differentiability implies continuity. In other words, if f has a derivative at $x = c$ then f is continuous at $x = c$.

The converse of this theorem is false! Continuity does NOT imply differentiability—see the corner, cusp, and vertical tangent examples above.

Example 3. At what points does the function graphed below fail to be differentiable?

§3.3—The Product and Quotient Rules

Differentiating products and quotients is not quite as simple as differentiating sums and differences. For example, consider writing x^5 as the product $x^3 \cdot x^2$. The product of the derivatives of the two factors in the second expression is $3x^2 \cdot 2x = 6x^3$, but we know that the derivative of this product is really $5x^4$. This shows that the derivative of a product is NOT equal to the product of the derivatives. Instead we have:

The Product Rule: $\dfrac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

Example 1. Find the derivative of each of the following functions.

(a) $h(x) = (x^2 + 3x + 1)e^x$

(b) $P(x) = (3x^{2/3} + 2e^x)(4 - x^{-5})$

Example 2. A company's revenue from t-shirt sales is given by $R(x) = xq(x)$, where $q(x)$ is the number of shirts it can sell at a price of \$$x$ apiece. If $q(10) = 200$ and $q'(10) = -13$, what is $R'(10)$?

Likewise, simple examples show that the derivative of a quotient is NOT equal to the quotient of the derivatives. The correct result is as follows:

The Quotient Rule: $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example 3. Find the derivative of each of the following functions.

(a) $h(x) = \dfrac{x}{x^5 + 3}$

(b) $F(x) = \dfrac{xe^x}{7 - \sqrt{x}}$

Example 4. Find the equation of the tangent line to the curve $y = \dfrac{e^x}{x+2}$ at the point $(0, 1/2)$.

§3.4—Rates of Change

Recall that the instantaneous rate of change of $y = f(x)$ with respect to x at $x = a$ is

$$\left.\frac{dy}{dx}\right|_{x=a} = f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists. This is the limit of the average rates of change of f over smaller and smaller intervals of the form $[a, a+h]$.

Some examples:

$s(t) =$ position $\quad s'(t) =$ velocity

$v(t) =$ velocity $\quad v'(t) =$ acceleration

$Q(t) =$ charge $\quad Q'(t) =$ current

$W(t) =$ work/energy $\quad W'(t) =$ power

$P(t) =$ population $\quad P'(t) =$ population growth rate

$R(x) =$ revenue from producing x units $\quad R'(x) =$ marginal revenue

$C(x) =$ cost of producing x units $\quad C'(x) =$ marginal cost

Example 1. The position (in meters) of a particle moving along the s-axis after t seconds is given by $s(t) = \frac{1}{3}t^3 - 2t^2 + 3t$ for $t \geq 0$.

(a) When is the particle moving forward? Backward?

(b) When is the particle's velocity increasing? Decreasing?

Example 2. A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/s reaches a height of $s = 24t - 0.8t^2$ meters in t seconds.

(a) Find the rock's velocity and acceleration at time t.

(b) How long does it take the rock to reach its highest point? What is its maximum height?

Example 3. Suppose that the cost of producing x washing machines is $C(x) = 2000 + 100x - 0.1x^2$.

(a) Find the marginal cost when 100 washing machines are produced.

(b) Compare the answer to (a) with the cost of producing the 101st machine.

§3.5—Higher Derivatives

When we compute the derivative of a function $f(x)$, we get a new function $f'(x)$. If we take the derivative of the function $f'(x)$, we get another new function, which is called the **second derivative** of $f(x)$ and denoted $f''(x)$. For example, the derivative of position with respect to time is velocity, and the derivative of velocity with respect to time is acceleration; therefore we say that acceleration is the second derivative of position: $a(t) = v'(t) = s''(t)$. We can continue this process to get higher derivatives:

$$f'(x) = \frac{d}{dx}[f(x)] = \frac{dy}{dx} \qquad \text{(1st derivative)}$$

$$f''(x) = \frac{d}{dx}[f'(x)] = \frac{d^2y}{dx^2} \qquad \text{(2nd derivative)}$$

$$f'''(x) = \frac{d}{dx}[f''(x)] = \frac{d^3y}{dx^3} \qquad \text{(3rd derivative)}$$

$$f^{(4)}(x) = \frac{d}{dx}[f'''(x)] = \frac{d^4y}{dx^4} \qquad \text{(4th derivative)}$$

and so on.

Example 1. Find the first, second, third, and fourth derivatives of the function $f(x) = x^{10} - 5x^4 + 3x + 2$.

Example 2. Find $f''(1)$ for the function $f(x) = x^3e^x$.

§3.6—Derivatives of Trigonometric Functions

Our goal in this section is to find formulas for the derivatives of the 6 basic trig functions:

$$\frac{d}{dx}(\sin x) = \qquad\qquad \frac{d}{dx}(\cot x) =$$
$$\frac{d}{dx}(\cos x) = \qquad\qquad \frac{d}{dx}(\sec x) =$$
$$\frac{d}{dx}(\tan x) = \qquad\qquad \frac{d}{dx}(\csc x) =$$

To find a formula for the derivative of the function $f(x) = \sin x$ we must return to the definition of the derivative in terms of a limit, which we studied in §3.2:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Here the trigonometric identity

$$\sin(x+h) = \sin x \cos h + \cos x \sin h$$

will help us get started, and we will need to recall two special trigonometric limits that we calculated back in §2.6:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

Example 1. Find the derivatives of the following functions.

(a) $f(x) = x^2 \sin x + 2 \cos x$

(b) $g(t) = \dfrac{e^t}{t - \sin t}$

Example 2. Use the quotient rule to find the derivatives of $\tan x$ and $\sec x$.

Example 3. Find the derivatives of the following functions.

(a) $f(x) = e^x \sec x + 2\tan x$

(b) $r(\theta) = \dfrac{1 + \cot\theta}{\theta^3 - 4\csc\theta}$

§3.7—The Chain Rule

How do we differentiate compositions of functions like e^{2x}, $\cos(x^2)$, $\sqrt{x^3+1}$, or $\sin^4 x$?

Suppose that $y = f(u)$ and $u = g(x)$, so that $y = f(g(x))$. It is helpful to think of f as the "outer" function and g as the "inner" function.

If we have $\dfrac{du}{dx} = g'(x) = 2$ and $\dfrac{dy}{du} = f'(u) = 3$, then a 1 unit change in x gives approximately 2 units change in u, which then gives approximately 6 units change in y.

This heuristic argument suggests that $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = f'(u)g'(x) = f'(g(x))g'(x)$. This is in fact true whenever f and g are differentiable and is known as the Chain Rule.

The Chain Rule: $\dfrac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

In words, this says that the derivative of a composition is the derivative of the outer function, evaluated at the inner function, times the derivative of the inner function.

Example 1. Find the derivatives of the following functions.

(a) $h(x) = e^{2x}$

(b) $h(x) = \cos(x^2)$

(c) $h(x) = \sqrt{x^3 + 1}$

(d) $h(x) = \sin^4 x$

Example 2. Find formulas for the velocity and acceleration of a particle whose position is given by $s(t) = 5\cos(2t)$.

In many problems, the Chain Rule must be applied in combination with other rules such as the product and quotient rules. It is also possible to have a composition within a composition, $f(g(h(x)))$, which requires more than one application of the Chain Rule. The following examples illustrate these more challenging situations.

Example 3. Find $\dfrac{dy}{dx}$ for the following functions.

(a) $y = e^{x^2} \cos 3x$

(b) $y = \sin(\sqrt{x^4 + 1})$

(c) $y = \dfrac{\tan(e^{2x})}{(x^2 + 1)^6}$

(d) $y = \sqrt{1 + \sqrt{1 + \sqrt{x}}}$

(e) $y = \cos^5(\sin^3 x)$

§3.8—Implicit Differentiation

Example 1. Find the equation of the tangent line to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.

Solution #1 (Solving for y):

Solution #2 (Differentiating implicitly):

In many of our examples it will not be possible to solve for y, so we'll be forced to use the second method.

Basic procedure for implicit differentiation:

1. Take the derivative of both sides with respect to x. In doing this, we think of y as a function of x, so derivatives of expressions involving y require the Chain Rule.

2. Solve algebraically for $\frac{dy}{dx}$ by collecting all the terms containing $\frac{dy}{dx}$ on one side of the equation and then factoring and dividing.

Remark. Implicit differentiation can be used to prove the power rule for rational exponents once it has been proved for integer exponents. For instance, if $y = x^{2/3}$, then we can write $y^3 = x^2$ and hence

$$3y^2 \frac{dy}{dx} = 2x \implies \frac{dy}{dx} = \frac{2x}{3y^2} = \frac{2x}{3x^{4/3}} = \frac{2}{3}x^{-1/3}.$$

Example 2. Find the slope of the tangent line to the curve $3x^4y^2 - 7xy^3 = 4 - 8y$ at the point $(0, 1/2)$.

Example 3. Find $\frac{dy}{dx}$ for the curve $x \cos y + y \cos x = 1$.

§3.9—Derivatives of Inverse Functions

Review of Inverses. A function is **one-to-one** if no y value occurs for two different values of x. For example, $f(x) = x^3$ is one-to-one, but $f(x) = x^2$ is not. This definition is captured graphically by the Horizontal Line Test: a function is one-to-one if and only if no horizontal line intersects its graph more than once. If f is one-to-one, then there is an **inverse** function f^{-1} defined by

$$f^{-1}(y) = x \iff f(x) = y.$$

The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f. The graph of f^{-1} is the reflection of the graph of f across the line $y = x$. Note that f and f^{-1} "undo" each other, meaning that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Warning: $f^{-1}(x)$ is NOT the same as $\dfrac{1}{f(x)}$.

Example 1. Find the inverse of the function $f(x) = 2x + 1$.

The Inverse Trig Functions. Even though the trigonometric functions are not one-to-one, we can define inverses for them by restricting their domains to intervals on which the functions are one-to-one. For example, $\sin x$ is one-to-one on the interval $-\pi/2 \le x \le \pi/2$ and $\cos x$ is one-to-one on the interval $0 \le x \le \pi$. Moreover, these functions cover the full range of y values between -1 and 1 as x runs over these restricted intervals. It is often helpful to think of the values of inverse trig functions as **angles**.

- $y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$
- $y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$
- $y = \tan^{-1} x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$
- $y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$
- $y = \sec^{-1} x$ is the number in $[0, \pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x$
- $y = \csc^{-1} x$ is the number in $[-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x$

The inverse trig functions are sometimes denoted by $\arcsin x$, $\arccos x$, $\arctan x$, *etc.*

Example 2. Evaluate each of the following.

(a) $\sin^{-1}(\frac{1}{2})$ (b) $\cos^{-1}(-\frac{1}{2})$ (c) $\tan^{-1}(1)$

Example 3. Convert $\cos\big(\tan^{-1}(\frac{x}{3})\big)$ to an algebraic expression in x.

The Derivative Formulas:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{x^2+1} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

The derivatives of the inverse "co" functions are just the negatives of these. For instance, $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \implies \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \implies (\cos^{-1} x)' = -(\sin^{-1} x)'$.

Example 4. Find the derivatives of the following functions.

(a) $y = (\sin^{-1} x)^3 + \cos^{-1}(x^3)$

(b) $y = x^2 \tan^{-1} x + \sqrt{\sec^{-1} x}$

Derivatives of inverse functions in general. If f is one-to-one and we write $g = f^{-1}$, then we have $f(g(x)) = x$, so differentiating both sides using the Chain Rule gives

$$f'(g(x))g'(x) = 1 \implies g'(x) = \frac{1}{f'(g(x))}.$$

For instance, in Example 1 we have $f(x) = 2x + 1$ and $g(x) = \frac{1}{2}x - \frac{1}{2}$, so $f'(x) = 2$ implies that $g'(x) = 1/f'(g(x)) = 1/2$. This is exactly the technique we used for $\sin^{-1} x$ above, and we will use it again in the next section to find derivatives of logarithmic functions.

§3.10—Derivatives of General Exponential and Logarithmic Functions

Review of logarithms. Suppose that $b > 0$ and $b \neq 1$. The function $y = b^x$ is one-to-one, so it has an inverse, namely $f^{-1}(x) = \log_b x$. The domain of $\log_b x$ is $(0, \infty)$, and the range is $(-\infty, \infty)$. Thus we have

$$y = \log_b x \iff b^y = x.$$

In other words, $\log_b x$ is the power that we must raise b to in order to get x. In particular, we have $b^{\log_b x} = x$ for all $x > 0$ and $\log_b b^x = x$ for all x. There are 3 main algebraic properties of logs to remember:

(1) $\log_b(xy) = \log_b x + \log_b y$ (2) $\log_b(x/y) = \log_b x - \log_b y$ (3) $\log_b x^r = r \log_b x$

The case where the base b is $e \approx 2.71828$ occurs so frequently that we use the special notation $\ln x$ to stand for $\log_e x$, so that $y = \ln x \iff e^y = x$.

Derivatives of Logarithmic Functions. We already know how to differentiate e^x, and we can use this to find the derivative of $\ln x$ via implicit differentiation:

Example 1. Compute the derivatives of the following functions.

(a) $y = x^3 \ln x + \ln(\cos x)$

(b) $y = (\ln x)^7 + \ln(\ln(\ln x))$

Remark. By writing $x^n = e^{\ln(x^n)} = e^{n \ln x}$ we can use the Chain Rule and the formula for the derivative of $\ln x$ to prove the power rule for any real exponent n:

$$\frac{d}{dx}(x^n) = \frac{d}{dx}(e^{n \ln x}) = e^{n \ln x} \cdot \frac{n}{x} = x^n \cdot \frac{n}{x} = nx^{n-1}.$$

Example 2. Find the derivatives of the following functions.

(a) $y = 2^x$

(b) $y = \log_2 x$

The calculations in Example 2 generalize to show that

$$\frac{d}{dx}(b^x) = (\ln b)b^x \qquad \text{and} \qquad \frac{d}{dx}(\log_b x) = \frac{1}{(\ln b)x}$$

whenever $b \neq 1$ is a positive constant. Note that the formulas for the derivative of e^x and $\ln x$ are special cases of this, since $\ln e = 1$.

Example 3. Find the derivatives of the following functions.

(a) $y = \sec(3^x \log_{10} x)$

(b) $y = 5^{\sin x} + \log_5(\tan^{-1}(x^5))$

Example 4 (Logarithmic differentiation). Find the derivative of the function $y = x^x$ by first taking the natural log of both sides and then differentiating implicitly.

§4.1—Linear Approximation and Applications

The tangent line to the curve $y = f(x)$ at $x = a$ is given by $y - f(a) = f'(a)(x - a)$, or

$$y = f(a) + f'(a)(x - a).$$

Thus when x is close to a we have the approximation $f(x) \approx f(a) + f'(a)(x - a)$.

This is called the **linear approximation** or **tangent line approximation** to f at a. The function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at $x = a$.

Example 1. Let $f(x) = \sqrt{x}$ and $a = 25$.

(a) Find the equation of the tangent line to f at $x = 25$, and write down the linearization.

(b) Use the linearization from part (a) to estimate $\sqrt{26}$, $\sqrt{23}$, and $\sqrt{28}$.

(c) Analyze the quality of the approximations from (b) by completing the following table.

x	Linear approx $L(x)$	$f(x) = \sqrt{x}$ via calculator	Error $= \lvert f(x) - L(x) \rvert$
26			
23			
28			

Example 2. Use a linear approximation to estimate each of the following:

(a) $\sin(0.02)$

(b) $\sqrt[3]{8.06}$

(c) $e^{0.03}$

(d) $\ln(0.95)$

When we are primarily interested in estimating the *change* in a given quantity, it is sometimes more convenient to rewrite the linear approximation in the form

$$f(x) - f(a) \approx f'(a)(x - a) \qquad \text{or} \qquad \Delta f \approx f'(a)\Delta x,$$

where $\Delta x = x - a$ and $\Delta f = f(x) - f(a)$. A typical application involves the analysis of error propagation, as in the following example.

Example 3. The edge of a cube is measured at 30 cm, with a possible error of ± 0.1 cm. Estimate the maximum possible error in computing the cube's volume.

§3.11—Related Rates

Suppose that two or more quantities are related by some equation. For instance, if C is the circumference of a circle and r is the radius, then $C = 2\pi r$. As another example, if a and b are the legs of a right triangle with hypotenuse c, then $a^2 + b^2 = c^2$.

If the quantities involved change with time, then we can differentiate both sides of the equation with respect to t to derive a relationship between the rates of change:

$$e.g. \qquad \frac{dC}{dt} = 2\pi\frac{dr}{dt} \qquad \text{or} \qquad 2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}.$$

If some of these rates of change are known, then we may be able to use these equations to solve for the unknown rates of change.

Example 1. The radius of a circular oil spill is increasing at a constant rate of 1.5 meters per second. How fast is the area of the spill increasing when the radius is 30 meters?

Example 2. Boyle's Law states that when a sample of gas is compressed at constant temperature the product of the pressure and the volume remains constant. At a certain instant, the volume of a gas is 600 cubic centimeters, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa per minute. How fast is the volume decreasing at this instant?

Example 3. A ladder 25 feet long is leaning against a vertical wall. The bottom of the ladder is being pulled horizontally away from the wall at a constant rate of 3 feet per second. At the instant when the bottom of the ladder is 15 feet from the wall, determine

(a) how fast the top of the ladder is sliding down the wall

(b) how fast the angle between the top of the ladder and the wall is changing

Example 4. A tank has the shape of an inverted cone with height 16 meters and base radius 4 meters. Water is being pumped into the tank at a constant rate of 2 cubic meters per minute. How fast is the water level rising when the water is 5 meters deep?

§4.2—Extreme Values

Let f be a function with defined on some interval I. We say that $f(a)$ is the **absolute maximum** of f on I if $f(a) \geq f(x)$ for all x in I. We say that $f(a)$ is the **absolute minimum** of f on I if $f(a) \leq f(x)$ for all x in I.

Note: The absolute maximum and minimum *values* refer to the largest and smallest y values on the graph, not the x values at which they occur.

We say that f has a **local maximum** at $x = c$ if $f(c) \geq f(x)$ for all x in some open interval containing c. We say that f has a **local minimum** at c if $f(c) \leq f(x)$ for all x in some open interval containing c.

Maxima and minima are sometimes called *extrema*. Absolute extrema are sometimes called *global* extrema, and local extrema are sometimes called *relative* extrema.

Example 1. Identify the coordinates of all absolute and local extrema for the function graphed below on the interval $[0, 10]$.

Example 2. Determine the absolute extrema of each function on the given intervals.

(a) $y = x^2$

(i) $[0, 2]$

(ii) $[0, 2)$

(iii) $(0, 2]$

(b) $y = 1/x$

(i) $(0, 3]$

(ii) $[3, \infty)$

(iii) $[-3, 3]$

Extreme Value Theorem. If f is continuous on the closed interval $[a, b]$, then f attains both an absolute maximum and an absolute minimum value in $[a, b]$.

A number c in the domain of f for which $f'(c) = 0$ or $f'(c)$ does not exist is called a **critical point** of f. The only *possible* places where local and absolute extrema occur are at critical points or at endpoints of the domain.

Example 3. Find the absolute maximum and minimum values of the function $f(x) = x^3 - 12x + 1$ on the interval $[-3, 5]$.

Example 4. Find the absolute maximum and minimum values of the function $f(x) = x^{5/3} - 10x^{2/3}$ on the interval $[-8, 8]$.

Example 5. Find the absolute maximum and minimum values of the function $f(x) = x^2 \ln x$ on the interval $[\frac{1}{2}, \infty)$.

§4.3—The Mean Value Theorem and Monotonicity

The Mean Value Theorem. Suppose that $y = f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one point c in (a, b) for which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

The picture that makes this "obvious":

Consequences of the MVT. One useful interpretation of the theorem is that there is some point in the interval at which the instantaneous rate of change is equal to the average rate of change. Another important consequence is the following:

Increasing/Decreasing Test:

If $f'(x) > 0$ for all x in some interval, then f is increasing on that interval (*i.e.*, the y values are getting larger).

If $f'(x) < 0$ for all x in some interval, then f is decreasing on that interval (*i.e.*, the y values are getting smaller).

Example 1. Show that the function $f(x) = x^3 + 2x + 4$ is always increasing.

Example 2. For what values of x is the function $f(x) = xe^{-x}$ decreasing?

First Derivative Test for Local Extrema: Suppose that c is a critical point of a differentiable function f.

1. If f' changes from negative to positive at c, then $f(c)$ is a local minimum.

2. If f' changes from positive to negative at c, then $f(c)$ is a local maximum.

3. If f' does not change sign at c, then $f(c)$ is neither a local maximum nor a local minimum.

Example 3. Find the intervals on which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and decreasing, and identify all local extrema of the function.

Example 4. Find the intervals on which the function $f(x) = x^5 - 15x^3 + 4$ is increasing and decreasing, and identify all local extrema of the function.

§4.4—The Shape of a Graph

Concavity

f is concave up $\iff$ f' is increasing $\iff$ $f'' > 0$
f is concave down $\iff$ f' is decreasing $\iff$ $f'' < 0$

Four basic shapes of graphs:

(a) $f' > 0,\ f'' > 0$ (b) $f' > 0,\ f'' < 0$ (c) $f' < 0,\ f'' > 0$ (d) $f' < 0,\ f'' < 0$

A point on the graph of f where the concavity changes is called an **inflection point** of f. These can only occur where $f'' = 0$ or f'' is undefined. Note that these are the local maxima and minima of f'.

Example 1. Find the points of inflection of $f(x) = x^3 - 6x^2 + 1$, and determine the intervals on which the curve is concave up and concave down.

Second Derivative Test for Local Extrema: Suppose f'' is continuous and $f'(c) = 0$.

1. If $f''(c) > 0$, then f has a local minimum at $x = c$.
2. If $f''(c) < 0$, then f has a local maximum at $x = c$.
3. If $f''(c) = 0$, then the test gives no information. In this case, we must go back to the first derivative test.

   e.g. $f(x) = x^3$ versus $f(x) = x^4$

Example 2. Use the second derivative test to determine the location of all local maxima and local minima of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$. [Compare with Example 3 in §4.3.]

§4.5—Graph Sketching and Asymptotes

Example 1. Sketch the graph of each of the following functions.

(a) $f(x) = x^4 - 4x^3$

(b) $f(x) = x^{2/3}(x - 5)$

Asymptotic behavior. We say that $\lim_{x\to\infty} f(x) = L$ if $f(x)$ can be made as close as we like to L by taking x sufficiently large. Similarly, we say that $\lim_{x\to-\infty} f(x) = L$ if $f(x)$ can be made as close as we like to L by taking $-x$ sufficiently large (that is, $|x|$ sufficiently large and $x < 0$). If

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L,$$

then the line $y = L$ is called a **horizontal asymptote** for the graph of $y = f(x)$.

Example 2. Evaluate each of the following limits.

(a) $\lim_{x\to\infty} \dfrac{8x^3 + 5x + 1}{2x^3 + 4}$ (b) $\lim_{x\to-\infty} \dfrac{3x^2 - 10}{x^5 + 3x + 1}$

Example 3. Sketch the graph of the rational function $f(x) = \dfrac{x+2}{x+1}$.

§4.6—Applied Optimization

Example 1. A farmer with 600 feet of fencing wants to construct a rectangular pen and then divide it in half with a fence parallel to one of the sides. What dimensions maximize the area of the pen?

Example 2. You are asked to design a cylindrical can (with top and bottom) of volume 500 cubic centimeters. What dimensions should the can have in order to minimize the amount of metal used?

When arguing that a critical number actually yields the optimal result, we frequently make use of the following principle:

First Derivative Test for Absolute Extrema. Suppose that f is continuous and that c is the *only* critical number of f. If $f(c)$ is a local maximum (resp. minimum), then it is also the absolute maximum (resp. minimum).

Example 3. You are asked to design an athletic complex in the shape of a rectangle with semi-circular ends. A running track 400 meters long is to go around the perimeter. What dimensions will give the rectangular playing field in the center the largest area?

Example 4. A box with no top is to have volume 4 cubic meters, and its base is to be a rectangle twice as long as it is wide. If the material for the bottom costs \$3 per square meter and the material for the sides costs \$1.50 per square meter, find the dimensions that minimize the total cost of constructing the box.

Example 5. You are standing on a sidewalk at the corner of a muddy rectangular field of length 1 mile and width 0.2 miles. You can run along the sidewalk bordering the long side of the field at 8 mph, and you can run through the mud at 5 mph. Assuming there is no sidewalk along the short side of the field, find the quickest route to the opposite corner.

Example 6. Find the volume of the largest cylinder that can be inscribed in a sphere of radius R. What percentage of the sphere's volume is occupied by such a cylinder?

§4.7—L'Hôpital's Rule

A general method for evaluating "0/0" or "∞/∞" type limits:

L'Hôpital's Rule. Suppose that either

(i) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ or (ii) $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$.

Then we have $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$. Here a can be a real number or $\pm\infty$.

Example 1. Evaluate the following limits.

(a) $\lim_{x\to 0} \dfrac{e^x - 1}{x}$

(b) $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$

Warning: L'Hôpital's Rule does not apply unless (i) or (ii) holds. For example,

$$0 = \lim_{x\to 0} \frac{\sin x}{x+1} \neq \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Sometimes it's necessary to apply l'Hôpital's Rule more than once:

Example 2. Evaluate the following limits.

(a) $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2}$

(b) $\lim_{x\to\infty} \dfrac{e^x}{x^4}$

We can sometimes deal with other indeterminate forms like $0 \cdot \infty$, 0^0, $\infty - \infty$, and 1^∞ by converting them to $0/0$ or ∞/∞ and then applying l'Hôpital's Rule.

Example 3. Evaluate the following limits.

(a) $\lim_{x \to 0^+} x \ln x$

(b) $\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$

(c) $\lim_{x \to \infty} \left(1 + \frac{1}{x} \right)^x$

§4.9—Antiderivatives

We say that F is an **antiderivative** of f if $F'(x) = f(x)$ for all x. For example,

x^2 and $x^2 + 1$ are antiderivatives of $2x$

$\sin x$ and $\sin x - 17$ are antiderivatives of $\cos x$

If F is any antiderivative of f, then it follows from the Mean Value Theorem that the most general antiderivative of f is $F(x) + C$, where C is an arbitrary constant. The set of all antiderivatives of f is denoted $\int f(x)\,dx$ and is called the **indefinite integral** of f with respect to x. For example,

$$\int 2x\,dx = x^2 + C \qquad \text{and} \qquad \int \cos x\,dx = \sin x + C.$$

Example 1. Evaluate the following indefinite integrals.

(a) $\displaystyle\int (x^2 + 2\cos x)\,dx$

(b) $\displaystyle\int (\sin x + x^{-6})\,dx$

(c) $\displaystyle\int (e^{3x} + 3\sec^2 x)\,dx$

Some Useful Indefinite Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad\qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin kx\,dx = -\frac{1}{k}\cos kx + C \qquad \int \cos kx\,dx = \frac{1}{k}\sin kx + C \qquad \int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$$

Example 2. Evaluate $\displaystyle\int \left(\cos 2x - 5\sqrt{x} + \frac{7}{x} + \frac{1}{\sqrt{1-x^2}}\right) dx.$

Note that we can split up sums and differences of indefinite integrals:

$$\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

However, there is no such law for products:

$$\int f(x)g(x)\, dx \neq \int f(x)\, dx \int g(x)\, dx.$$

For example,

$$\int x \cos x\, dx \neq \frac{x^2}{2} \sin x + C.$$

Initial Conditions. If we know a function's derivative *and* the value of the function at one point, we can determine the function by first finding the general antiderivative and then using the known value to solve for C.

Example 3. Suppose that $f'(x) = 3x^2$ and $f(1) = 5$. Find a formula for $f(x)$.

Example 4. A particle's acceleration is given by $a(t) = 5 + 4t - 2t^2$, and its initial velocity and position are $v(0) = 3$ and $s(0) = 10$. Find formulas for $v(t)$ and $s(t)$.

§5.1—Approximating and Computing Area

Example 1. Estimate the area of the region bounded by the curve $y = x^2$ and the x-axis between $x = 0$ and $x = 2$ by approximating the region with 4 rectangles of equal width whose heights are determined using

(a) left endpoints

(b) right endpoints

(c) midpoints

Using a larger number of rectangles gives a better estimate of the area, and we define the exact area to be the limit of these approximations as the number of rectangles tends to infinity. In order to add up a large number of terms, it is convenient to use **sigma notation**:

$$\sum_{j=1}^{N} a_j = a_1 + a_2 + a_3 + \cdots + a_{N-1} + a_N$$

Example 2. Evaluate the following:

(a) $\sum_{j=1}^{7} j$

(b) $\sum_{j=1}^{5} j^2$

Example 3. Use sigma notation to write the right-endpoint approximation R_N for the area of the region bounded by the curve $y = x^2$ and the x-axis between $x = 0$ and $x = 2$.

To find the exact area under the curve, we need to find a way to express R_N (or L_N or M_N) in closed form so that we can compute the limit as $N \to \infty$. In general, it is quite difficult to do this, but there are many special cases that can be handled; for instance:

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2} \qquad \text{and} \qquad \sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6}.$$

Example 4. Calculate the exact area of the region bounded by the curve $y = x^2$ and the x-axis between $x = 0$ and $x = 2$.

Finding distance traveled. By applying the same reasoning as above and using the fact that distance = velocity $\times$ time when velocity is constant, we see that the net change in position of an object over an interval is the area under its velocity curve.

Example 5. A car's velocity during a 1-hour period is measured at 12-minute intervals:

time (hours)	0	0.2	0.4	0.6	0.8	1.0
velocity (miles per hour)	66	75	78	82	79	74

Estimate the total distance traveled by the car during the hour using

(a) left endpoints

(b) right endpoints

§5.2—The Definite Integral

To approximate the area bounded by a continuous function $y = f(x)$ and the x-axis on the interval $[a, b]$, we divide into N subintervals of width

$$\Delta x = \frac{b-a}{N}.$$

The jth subinterval is the interval $[x_{j-1}, x_j]$, where $x_j = a + j\Delta x$. For each j, we use a rectangle of height $f(x_j)$ and width Δx to approximate the area under that portion of the curve.

The **Riemann sum**

$$R_N = \sum_{j=1}^{N} f(x_j)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_N)\Delta x$$

approximates the total area under the curve on the interval $[a, b]$. We get the exact area by letting $N \to \infty$, which gives the **definite integral** of f from a to b:

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{j=1}^{N} f(x_j)\Delta x.$$

Here the function f is called the **integrand** and the numbers a and b are the **limits of integration**. Note that the choice to use right-endpoints here is simply a convenience; for continuous functions, we could choose points randomly in each subinterval and still get the same result as $\Delta x \to 0$.

Example 1. Calculate the following definite integrals directly from the definition.

(a) $\displaystyle\int_0^3 x\,dx$

(b) $\int_0^b x^2\,dx$

If $f(x)$ takes both positive and negative values on $[a, b]$, then the definite integral gives the "signed area" under the curve. That is, areas above the x-axis are counted positively, and areas below the x-axis are counted negatively.

Example 2. Evaluate the following integrals.

(a) $\int_0^{2\pi} \sin x\,dx$

(b) $\int_{-2}^{3} x\,dx$

Properties of the definite integral

- Conventions: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ and $\int_a^a f(x)\,dx = 0$
- Linearity: $\int_a^b (kf(x) \pm mg(x))\,dx = k\int_a^b f(x)\,dx \pm m\int_a^b g(x)\,dx$ (k, m constant)
- Additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- Comparison: If $f(x) \leq g(x)$ for all x in $[a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

Example 3. Suppose that $\int_0^1 f(x)\,dx = 2$ and that $f(x) \leq 4$ for all x in $[1, 3]$. What is the largest possible value that the integral $\int_0^3 f(x)\,dx$ could have?

§5.3—The Fundamental Theorem of Calculus, Part I

It turns out that the key to evaluating definite integrals efficiently is finding an antiderivative for the integrand. We actually observed a special case of this in Example 5 of Section 5.1 when we saw that the area under a velocity graph gives the net change in position. More generally, if f has an antiderivative F, then we can view f as a rate of change of F and apply the same reasoning to establish the following:

Fundamental Theorem of Calculus, Part I. If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This theorem (often called the FTC for short) may be interpreted as saying that the definite integral of a rate of change gives the total change. For example, if f represents velocity and F represents position, then the definite integral of velocity is change in position.

Example 1. Use the FTC to calculate each of the following.

(a) $\displaystyle\int_0^2 x^2\,dx$

(b) $\displaystyle\int_0^{\pi/2} \cos x\,dx$

(c) $\displaystyle\int_0^1 \frac{dx}{1+x^2}$

Example 2. Evaluate each of the following definite integrals.

(a) $\displaystyle\int_0^3 e^{5x}\,dx$ (b) $\displaystyle\int_0^1 \sqrt{x}(x^2+3)\,dx$

Example 3. Find the area bounded by the curve $y = 1/x$ and the x-axis between $x = 2$ and $x = 6$.

Example 4. What is wrong with the following calculation?

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}.$$

Example 5. Evaluate $\displaystyle\int_0^5 \sqrt{25-x^2}\,dx$.

§5.4—The Fundamental Theorem of Calculus, Part II

Example 1. Consider the function f graphed below, and let $A(x)$ denote the signed area under the curve on the interval $[0, x]$. Calculate each of the following.

(a) $A(0)$

(b) $A(2)$

(c) $A(5)$

Fundamental Theorem of Calculus, Part II. Suppose that f is continuous on $[a, b]$, and let $A(x) = \int_a^x f(t)\,dt$. Then

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

That is, $A(x)$ is the antiderivative of $f(x)$ satisfying the initial condition $A(a) = 0$.

Why is this true?

Interpretation: Differentiation and integration are "inverse" operations, *i.e.*, $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.

Example 2. Calculate each of the following derivatives.

(a) $\frac{d}{dx}\int_1^x \frac{\sin t}{t}\,dt$

(b) $\frac{d}{dx}\int_{-5}^x t^3 e^t\,dt$

Example 3. Find the derivative of each function.

(a) $F(x) = \int_4^{x^3} \sqrt{1+t^2}\, dt$

(b) $G(x) = \int_{e^{2x}}^{10} \cos^3 t\, dt$

Example 4. For what values of x is the function $F(x) = \int_0^x \frac{1}{1+t+t^2}\, dt$ concave up?

Example 5. Find a function $F(x)$ such that $F'(x) = \ln x$ and $F(1) = 3$.

§5.5—Net or Total Change as the Integral of a Rate

A useful interpretation of the FTC (Part I) is that the definite integral of a rate of change gives total change. For instance, if $s(t)$ represents position, then $s'(t)$ is velocity (or rate of change of position), and we have

$$\int_a^b s'(t)\,dt = s(b) - s(a).$$

That is, the definite integral of velocity gives the net change in position. The same principle applies when integrating any function that can be viewed as a rate of change.

Example 1. A particle's velocity is given by $v(t) = t^2 - 4t + 3$.

(a) Find the object's net change in position over the interval $0 \leq t \leq 3$.

(b) Find the total distance traveled by the object over the interval $0 \leq t \leq 3$.

Example 2. The rate of energy consumption in a certain home (in kilowatts) is modeled by the function $R(t) = 2 + 0.5\cos(\pi t/3)$, where t is measured in months since January 1. According to this model, how many kilowatt-hours of energy will be used in a typical year?

§5.6—The Substitution Method

In earlier sections, we obtained formulas like

$$\int \cos 2x\, dx = \frac{1}{2}\sin 2x + C \qquad \text{and} \qquad \int e^{5x}\, dx = \frac{1}{5}e^{5x} + C$$

by mentally attempting to reverse the effect of the chain rule. A more systematic approach is to substitute a new variable for the inner function. For instance, if we let $u = 2x$ in the first integral above, then $du = 2dx$, and thus $dx = \frac{1}{2}du$, so we get

$$\int \cos 2x\, dx = \int (\cos u)\frac{1}{2}du = \frac{1}{2}\int \cos u\, du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C.$$

In general, we can evaluate $\displaystyle\int f(g(x))g'(x)\, dx$ by substituting $u = g(x)$ and $du = g'(x)\, dx$.

Example 1. Evaluate the following indefinite integrals.

(a) $\displaystyle\int 2xe^{x^2}\, dx$

(b) $\displaystyle\int \sqrt{3x+4}\, dx$

(c) $\displaystyle\int x^4 \cos(x^5)\, dx$

Example 2. What is wrong with the following calculation of $\int \cos(x^5)\,dx$?

Let $u = x^5$, so that $du = 5x^4\,dx$. Then $dx = \dfrac{du}{5x^4}$, so

$$\int \cos(x^5)\,dx = \int (\cos u)\frac{du}{5x^4} = \frac{1}{5x^4}\int \cos u\,du = \frac{1}{5x^4}\sin u + C = \frac{\sin(x^5)}{5x^4} + C.$$

Example 3. Evaluate the following indefinite integrals.

(a) $\displaystyle\int \frac{(1+\ln x)^{10}}{x}\,dx$

(b) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

(c) $\displaystyle\int \frac{\sec^2 x}{1+\tan x}\,dx$

Substitution in Definite Integrals: $\displaystyle\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

Example 4. Use substitution to evaluate the following definite integrals.

(a) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

(b) $\displaystyle\int_0^{\pi/2} (1+\sin^3 x)\cos x\,dx$

(c) $\displaystyle\int_2^4 \frac{dx}{x\ln x}$

(d) $\displaystyle\int_{\pi/4}^{\pi/2} 3^{\cot\theta}\csc^2\theta\,d\theta$

§5.7—Further Transcendental Functions

We record here for reference two important integrals involving the inverse trig functions:

$$\int \frac{dx}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C \qquad \text{and} \qquad \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Example. Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{dx}{25 + x^2}$

(b) $\displaystyle\int \frac{dx}{\sqrt{4 - 9x^2}}$

(c) $\displaystyle\int_0^{1/2} \frac{x\, dx}{16x^4 + 1}$

(d) $\displaystyle\int_1^{\sqrt{e}} \frac{dx}{x\sqrt{1 - (\ln x)^2}}$