NEW INFINITE q-PRODUCT EXPANSIONS WITH VANISHING COEFFICIENTS

JAMES MC LAUGHLIN

Abstract. Motivated by results of Hirschhorn, Tang, and Baruah and Kaur on vanishing coefficients (in arithmetic progressions) in a new class of infinite product which have appeared recently, we further examine such infinite products, and find that many such results on vanishing coefficients may be grouped into families.

For example, one result proven in the present paper is that if $b \in \{1, 2, \ldots, 9, 10\}$ and the sequence $\{r_n\}$ is defined by

$$(q^{8b}, q^{11-8b}; q^{11})^3_\infty (q^{11-b}, q^{11+b}; q^{22})_\infty =: \sum_{n=-756}^{\infty} r_n q^n.$$

then $r_{11n+6b^2+b} = 0$ for all n. Further, if $b \in \{1, 3, 5, 7, 9\}$, then $r_{11n+4b^2+b} = 0$ for all n also. Each particular value of b gives a specific result, such as the following (for $b = 1$): if the sequences $\{a_n\}$ is defined by

$$\sum_{n=0}^{\infty} a_n q^n := (q^3, q^8; q^{11})^3_\infty (q^{10}, q^{12}; q^{22})_\infty,$$

then $a_{11n+5} = a_{11n+7} = 0$.

1. Introduction

In [4], Hirschhorn gave the first examples of a new class of infinite q-products which have the property that when the product is expanded as a series in q, then the coefficients in one or more arithmetic progressions vanish. More precisely, he proved the following.

Let the sequences $\{a_n\}$ and $\{b_n\}$ are defined by

$$\sum_{n=0}^{\infty} a_n q^n := (-q, -q^4; q^5)_\infty (q, q^9; q^{10})^3_\infty,$$

$$\sum_{n=0}^{\infty} b_n q^n := (-q^2, -q^3; q^5)_\infty (q^3, q^7; q^{10})^3_\infty.$$

Then $a_{5n+2} = a_{5n+4} = b_{5n+1} = b_{5n+4} = 0$.

Date: March 18, 2020.

2000 *Mathematics Subject Classification.* Primary:11B65. Secondary: 33D15, 05A19.

Key words and phrases. q-Series, Infinite Products, Infinite q-Products, Vanishing Coefficients .

Here we are using the standard notation

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$
$$(a_1,\dots,a_j;q)_\infty := (a_1;q)_\infty \cdots (a_j;q)_\infty.$$

Similar results were proven in [6] by Tang, who showed, amongst other results that if the sequences $\{a_2(n)\}$, $\{b_2(n)\}$, $\{a_3(n)\}$ and $\{b_3(n)\}$ are defined by

$$\sum_{n=0}^{\infty} a_2(n)q^n := (-q,-q^4;q^5)_\infty^3(q^2,q^8;q^{10})_\infty,$$
$$\sum_{n=0}^{\infty} b_2(n)q^n := (-q^2,-q^3;q^5)_\infty^3(q^4,q^6;q^{10})_\infty,$$
$$\sum_{n=0}^{\infty} a_3(n)q^n := (-q,-q^4;q^5)_\infty^3(q^3,q^7;q^{10})_\infty,$$
$$\sum_{n=0}^{\infty} b_3(n)q^n := (-q^2,-q^3;q^5)_\infty^3(q,q^9;q^{10})_\infty,$$

then $a_2(5n+4) = b_2(5n+1) = a_3(5n+3) = a_3(5n+4) = b_3(5n+3) = b_3(5n+4) = 0$. Tang's other results concern various infinite products of a slightly different format, where (infinite product)^3 above is replaced with (infinite product)^2, but infinite products in this format are not considered in the present paper.

In [1], Baruah and Kaur prove a number of similar results, including the following. Let the sequences $\{k_n\}$, $\{l_n\}$, $\{u_n\}$ and $\{v_n\}$ be defined by

$$\sum_{n=0}^{\infty} k_nq^n := (q,q^4;q^5)_\infty(q,q^9;q^{10})_\infty^3, \tag{1.1}$$
$$\sum_{n=0}^{\infty} l_nq^n := (q^2,q^3;q^5)_\infty(q^3,q^7;q^{10})_\infty^3,$$
$$\sum_{n=0}^{\infty} u_nq^n := (q,q^4;q^5)_\infty^3(q^3,q^7;q^{10})_\infty,$$
$$\sum_{n=0}^{\infty} v_nq^n := (q^2,q^3;q^5)_\infty^3(q,q^9;q^{10})_\infty.$$

Then $k_{5n+4} = l_{5n+4} = u_{5n+4} = v_{5n+3} = 0$. The authors also prove the results of Tang listed above.

At the end of the paper [6], Tang considers the more general problem of finding triples (r,s,t) such that if the sequences $\{a_{r,s,t}(n)\}$, $\{b_{r,s,t}(n)\}$ are defined by

$$\sum_{n=0}^{\infty} a_{r,s,t}(n)q^n := (-q^r, -q^{t-r}; q^t)_\infty^3 (q^s, q^{2t-s}; q^{2t})_\infty, \tag{1.2}$$
$$\sum_{n=0}^{\infty} b_{r,s,t}(n)q^n := (-q^r, -q^{t-r}; q^t)_\infty (q^s, q^{2t-s}; q^{2t})_\infty^3,$$

then these sequences vanish in one or more arithmetic progressions modulo t. Tang states a number of results without proof for $t = 7$ and $t = 11$, but remarks that they may be proved by the methods employed to prove the results in the paper.

In the present paper we consider products like those in (1.2), but with $(q^r, q^{t-r}; q^t)_\infty$ instead of $(-q^r, -q^{t-r}; q^t)_\infty$, and prove a number of results for $t = 5, 7$ and 11 (as Tang remarks, there appears to be no similar results for $t = 13$ or $t = 17$). In a later section we list a number of families of results like those in (1.2), and similar results in the situation where the negative sign is in the other infinite product, but do not give the proofs, since they follow by using the same methods used to prove Theorem 2.1 - Theorem 4.2.

Observe that multiplying the equations (1.2) by infinite products of the forms $(q^t; q^t)_\infty$ or $(q^{2t}; q^{2t})_\infty$ will not have any effect on coefficients that vanish in an arithmetic progression modulo t, and our method of proof involves multiplying the right sides of (1.2) by such products, so as to convert these right sides in products of Jacobi triple products. For space saving reasons we will frequently use the notation

$$\langle a; q^j \rangle_\infty$$

to represent the triple product $(a, q^j/a, q^j; q^j)_\infty$ more compactly.

The main tool used to deal the part of the product consisting of a Jacobi triple product cubed is the extended quintuple product formula (see Cao [2, Eq. (3.2)] or Mc Laughlin [5, Eq. (4.6)])

$$\begin{aligned}
\langle -qa; q^2 \rangle_\infty \langle -qb; q^2 \rangle_\infty \langle -qc; q^2 \rangle_\infty &= \left\langle -\frac{q^2 a}{c}; q^4 \right\rangle_\infty \\
&\Bigg\{ \left\langle -\frac{q^6 ac}{b^2}; q^{12} \right\rangle_\infty \langle -q^3 abc; q^6 \rangle_\infty + qb \left\langle -\frac{q^2 ac}{b^2}; q^{12} \right\rangle_\infty \langle -q^5 abc; q^6 \rangle_\infty \\
&\quad + q^4 b^2 \left\langle -\frac{ac}{q^2 b^2}; q^{12} \right\rangle_\infty \langle -q^7 abc; q^6 \rangle_\infty \Bigg\} + \frac{q^2 a}{b} \left\langle -\frac{q^4 a}{c}; q^4 \right\rangle_\infty \\
&\Bigg\{ \left\langle -\frac{q^{12} ac}{b^2}; q^{12} \right\rangle_\infty \langle -q^3 abc; q^6 \rangle_\infty + \frac{b}{q} \left\langle -\frac{q^8 ac}{b^2}; q^{12} \right\rangle_\infty \langle -q^5 abc; q^6 \rangle_\infty \\
&\quad + b^2 \left\langle -\frac{q^4 ac}{b^2}; q^{12} \right\rangle_\infty \langle -q^7 abc; q^6 \rangle_\infty \Bigg\}.
\end{aligned} \tag{1.3}$$

Observe that this product simplifies considerably when $a = b = c$.

For ease of use, we state two equivalent forms of the Jacobi triple product identity. Both forms are used frequently to expand a Jacobi triple as an infinite bilateral series or to go in the reverse direction, sometimes one form is used, sometimes the other, and it is simpler to have both forms available for easy reference.

$$\sum_{n=-\infty}^{\infty}(-z)^n q^{n^2}=(zq,q/z,q^2;q^2)_\infty. \tag{1.4}$$

$$\sum_{n=-\infty}^{\infty}(-z)^n q^{n(n-1)/2}=(z,q/z,q;q)_\infty. \tag{1.5}$$

We remark that one difference between the method used in the present paper and the methods in the papers referenced above is that our method allows all results in a given family to proved simultaneously (these families contain up to 10 separate results in the case $t=11$, for example). In contrast, the methods used in [1], [4] and [6] allow just one result to be proved at a time. It is not clear to the present author how easily the proofs in these papers may be adapted to deal with the cases $t=7$ and $t=11$, while in the present paper the method of proof is uniform for the cases $t=5,7$ and 11.

2. Mod 5

In this section we reprove the results (1.1) of Baruah and Kaur. Our method of proof is different from theirs, and it also gives an illustration of how the results are proved in families.

Theorem 2.1. *For $b\in\{1,2\}$ define the sequence $\{r_n\}$ by*

$$(q^b,q^{5-b};q^5)_\infty^3(q^{5-2b},q^{5+2b};q^{10})_\infty=:\sum_{n=0}^{\infty}r_nq^n. \tag{2.1}$$

Then $r_{5n+4b}=0$ for all n.

Proof. In (1.3), replace q with $q^{5/2}$ and set $a=b=c=-q^{5/2-b}$ to get that

$$\begin{aligned}&\left\langle q^b;q^5\right\rangle_\infty^3= \\ &\left\langle q^{3b};q^{15}\right\rangle_\infty\left\{\left\langle -q^5;q^{10}\right\rangle_\infty\left\langle -q^{15};q^{30}\right\rangle_\infty+q^5\left\langle -1;q^{10}\right\rangle_\infty\left\langle -1;q^{30}\right\rangle_\infty\right\} \\ &\quad+\left(q^{10-2b}\left\langle q^{-10+3b};q^{15}\right\rangle_\infty-q^{5-b}\left\langle q^{-5+3b};q^{15}\right\rangle_\infty\right) \\ &\quad\times\left\{\left\langle -1;q^{10}\right\rangle_\infty\left\langle -q^{10};q^{30}\right\rangle_\infty+\left\langle -q^5;q^{10}\right\rangle_\infty\left\langle -q^5;q^{30}\right\rangle_\infty\right\}.\end{aligned} \tag{2.2}$$

Multiply this equation across by $\left\langle q^{5+2b};q^{10}\right\rangle_\infty$ and isolate those terms in the series expansion with the powers of q that are congruent to $4b$ modulo 5.

Define

$$\left\langle q^{5+2b};q^{10}\right\rangle_\infty \left\langle q^{3b};q^{15}\right\rangle_\infty =: \sum_{n=0}^{\infty} u_n q^n, \qquad F_1(q) := \sum_{\substack{n=0\\ n\equiv 4b \pmod 5}}^{\infty} u_n q^n.$$

By (1.4) and (1.5),

(2.3)

$$\left\langle q^{5+2b};q^{10}\right\rangle_\infty \left\langle q^{3b};q^{15}\right\rangle_\infty = \sum_{m,n=-\infty}^{\infty} (-1)^{n+m} q^{5n^2+2bn+3bm+15m(m-1)/2}.$$

To get the terms in this series that are in $F_1(q)$, it is necessary and sufficient that $2bn+3bm \equiv 4b \mod 5$, or that $2n+3m \equiv 4 \mod 5$ and thus $-3n+3m \equiv 4 \mod 5$ also. Set $2n+3m = 4+5r$ and $-3n+3m = 4+5s$, from which it can be seen that s has the form $s = 3k+1$ for k an integer. Hence $n = r-3k-1$ and $m = 2+r+2k$, $(-1)^{n+m} = (-1)^{1+k}$,

$$5n^2+2bn+3bm+\frac{15m(m-1)}{2} = 20+4b+75k+75k^2+5br+\frac{25(r^2+r)}{2},$$

so that, by (1.4) and (1.5) once again,

$$F_1(q) = -q^{20+4b}\left\langle 1;q^{150}\right\rangle_\infty \left\langle -q^{-5b};q^{25}\right\rangle_\infty = 0.$$

Similarly, define

$$q^{10-2b}\left\langle q^{5+2b};q^{10}\right\rangle_\infty \left\langle q^{-10+3b};q^{15}\right\rangle_\infty =: \sum_{n=0}^{\infty} v_n q^n,$$

$$F_2(q) := \sum_{\substack{n=0\\ n\equiv 4b \pmod 5}}^{\infty} v_n q^n.$$

By (1.4) and (1.5) again,

$$\left\langle q^{5+2b};q^{10}\right\rangle_\infty \left\langle q^{-10+3b};q^{15}\right\rangle_\infty = \sum_{m,n=-\infty}^{\infty} (-1)^{n+m} q^{5n^2+2bn+(-10+3b)m+15m(m-1)/2} \tag{2.4}$$

Upon taking into consideration the factor q^{10-2b}, to get the terms that are in $F_2(q)$, it is necessary and sufficient that $2bn+3bm \equiv 6b \mod 5$, or that $2n+3m \equiv 6 \mod 5$. By similar reasoning to that employed for $F_1(q)$, it follows that m and n have the forms $m = 2+2k+r$, $n = r-3k$ for integers k and r, $(-1)^{n+m} = (-1)^k$,

$$5n^2+2bn+(-10+3b)m+\frac{15m(m-1)}{2} = -5+b+25k+75k^2+5br+\frac{25(r^2-r)}{2},$$

so that, by (1.4) and (1.5) once again,

$$F_2(q) = q^{5-b} \left\langle q^{50}; q^{150} \right\rangle_\infty \left\langle q^{5b}; q^{25} \right\rangle_\infty .$$

Finally, define

$$-q^{5-b} \left\langle q^{5+2b}; q^{10} \right\rangle_\infty \left\langle q^{-5+3b}; q^{15} \right\rangle_\infty =: \sum_{n=0}^{\infty} w_n q^n,$$

$$F_3(q) := \sum_{\substack{n=0 \\ n\equiv 4b \pmod 5}}^{\infty} w_n q^n.$$

Once more employing (1.4) and (1.5),

$$\left\langle q^{5+2b}; q^{10} \right\rangle_\infty \left\langle q^{-5+3b}; q^{15} \right\rangle_\infty = \sum_{m,n=-\infty}^{\infty} (-1)^{n+m} q^{5n^2+2bn+(-5+3b)m+15m(n-1)/2} \tag{2.5}$$

This time, to get the terms that are in $F_3(q)$, it is necessary and sufficient (after taking account of the q^{5-b} factor) that $2bn+3bm \equiv 5b \equiv 0 \mod 5$, or that $2n + 3m \equiv 0 \mod 5$. By similar reasoning to that employed for $F_1(q)$ and $F_2(q)$, it follows that this time m and n have the forms $m = 2k + r$, $n = r - 3k$ for integers k and r, $(-1)^{n+m} = (-1)^k$,

$$5n^2 + 2bn + (-5 + 3b)m + \frac{15m(m-1)}{2} = -25k + 75k^2 + 5br + \frac{25(r^2 - r)}{2},$$

so that, by (1.4) and (1.5) once again,

$$F_3(q) = -q^{5-b} \left\langle q^{50}; q^{150} \right\rangle_\infty \left\langle q^{5b}; q^{25} \right\rangle_\infty = -F_2(q).$$

Since $F_1(q) = 0$ and $F_2(q) + F_3(q) = 0$, then $r_{5n+4b} = 0$ for all n as claimed, and the proof is complete. □

Corollary 2.1. *If the sequences $\{a_n\}$ and $\{b_n\}$ are defined by*

$$\sum_{n=0}^{\infty} a_n q^n := (q, q^4; q^5)_\infty^3 (q^3, q^7; q^{10})_\infty, \tag{2.6}$$

$$\sum_{n=0}^{\infty} b_n q^n := (q^2, q^3; q^5)_\infty^3 (q, q^9; q^{10})_\infty, \tag{2.7}$$

then $a_{5n+4} = b_{5n+3} = 0$.

Proof. These results are respectively, the cases $b = 1$ and $b = 2$ of Theorem 2.1. □

Theorem 2.2. *For $b \in \{1,2\}$ define the sequence $\{r_n\}$ by*

$$(q^{2b}, q^{5-2b}; q^5)_\infty (q^{5-2b}, q^{5+2b}; q^{10})^3_\infty =: \sum_{n=0}^{\infty} r_n q^n. \tag{2.8}$$

Then $r_{5n+3b^2+b} = 0$ for all n.

Proof. In (1.3), replace q with q^5 and set $a = b = c = -q^{2b}$ to get that

$$\begin{aligned}(2.9)\quad & \left\langle q^{5+2b}; q^{10}\right\rangle^3_\infty = \\ & \left\langle q^{15+6b}; q^{30}\right\rangle_\infty \left\{ \left\langle -q^{10}; q^{20}\right\rangle_\infty \left\langle -q^{30}; q^{60}\right\rangle_\infty + q^{10} \left\langle -1; q^{20}\right\rangle_\infty \left\langle -1; q^{60}\right\rangle_\infty \right\} \\ & \quad - \left(q^{5-2b} \left\langle q^{5+6b}; q^{30}\right\rangle_\infty + q^{5+2b} \left\langle q^{5-6b}; q^{30}\right\rangle_\infty \right) \\ & \quad \times \left\{ \left\langle -1; q^{20}\right\rangle_\infty \left\langle -q^{20}; q^{60}\right\rangle_\infty + \left\langle -q^{10}; q^{20}\right\rangle_\infty \left\langle -q^{10}; q^{60}\right\rangle_\infty \right\}.\end{aligned}$$

The remainder of the proof of Theorem 2.2 after the specialization of (1.3), like the proof of all of the remaining theorems, is similar to the proof of Theorem 2.1 after employing (1.3).

Multiply the last equation above across by $\left\langle q^{2b}; q^5\right\rangle_\infty$ and isolate those terms in the series expansion with the powers of q that are congruent to $3b^2 + b$ modulo 5.

Define

$$\left\langle q^{2b}; q^5\right\rangle_\infty \left\langle q^{15+6b}; q^{30}\right\rangle_\infty =: \sum_{n=0}^{\infty} u_n q^n, \qquad F_1(q) := \sum_{\substack{n=0 \\ n\equiv 3b^2+b \pmod 5}}^{\infty} u_n q^n.$$

By employing (1.4) and (1.5),

$$\left\langle q^{2b}; q^5\right\rangle_\infty \left\langle q^{15+6b}; q^{30}\right\rangle_\infty = \sum_{m,n=-\infty}^{\infty} (-1)^{n+m} q^{15n^2+6bn+2bm+5m(m-1)/2}.$$

To get the terms in this series that are in $F_1(q)$, it is necessary and sufficient that $6bn + 2bm \equiv 3b^2 + b \mod 5$, or that $6n + 2m \equiv 3b + 1 \mod 5$ and thus $6n - 3m \equiv 3b + 1 \mod 5$ also. Set $6n + 2m = 3b + 1 + 5r$ and $6n - 3m = 3b + 1 + 5s$. From the first of these equations it can be seen that r has the form $r = 2j + 1 + b$, and from the second equation that s has the form $s = 3k + 1$ where j and k are integers. Thus $m = r - s = 2j - 3k + b$, $n = 1 + b + j + k$, $(-1)^{n+m} = (-1)^{1+j}$,

$$\begin{aligned}15n^2 + 6bn + 2bm + \frac{5m(m-1)}{2} \\ = 15 + \frac{51b^2}{2} + \frac{67b}{2} + 25j + 50bj + 25j^2 + 15bk + \frac{75(k^2+k)}{2},\end{aligned}$$

so that, by (1.4) and (1.5) once again,

$$F_1(q) = -q^{15+51b^2/2+67b/2} \left\langle q^{-50b}; q^{50} \right\rangle_\infty \left\langle -q^{-15b}; q^{75} \right\rangle_\infty,$$

and $F_1(q) = 0$ for b a positive integer.

Similarly, define

$$q^{5-2b} \left\langle q^{2b}; q^5 \right\rangle_\infty \left\langle q^{5+6b}; q^{30} \right\rangle_\infty =: \sum_{n=0}^{\infty} v_n q^n,$$

$$F_2(q) := \sum_{\substack{n=0 \\ n \equiv 3b^2+b \pmod 5}}^{\infty} v_n q^n.$$

By (1.4) and (1.5) again,

$$\left\langle q^{2b}; q^5 \right\rangle_\infty \left\langle q^{5+6b}; q^{30} \right\rangle_\infty = \sum_{m,n=-\infty}^{\infty} (-1)^{n+m} q^{15n^2-10n+6bn+2bm+5m(m-1)/2}$$

Upon taking into consideration the factor q^{5-2b}, to get the terms that are in $F_2(q)$, it is necessary and sufficient that $6bn + 2bm \equiv 3b^2 + 3b \mod 5$, or that $6n + 2m \equiv 3b + 3 \mod 5$ and thus $6n - 3m \equiv 3b + 3 \mod 5$ also. Set $6n + 2m = 3b + 3 + 5r$ and $6n - 3m = 3b + 3 + 5s$. From the first of these equations it can be seen that r has the form $r = 2j + 1 + b$, and from the second equation that s has the form $s = 3k$ where j and k are integers. Thus $m = r - s = 2j + 1 + b - 3k$, $n = 1 + b + j + k$, $(-1)^{n+m} = (-1)^j$,

$$15n^2 - 10n + 6bn + 2bm + \frac{5m(m-1)}{2} = 5 + \frac{51b^2}{2} + \frac{61b}{2} + 25j + 50bj + 25j^2 + 15bk + \frac{75k^2 + 25k}{2},$$

so that, by (1.4) and (1.5) once again,

$$F_2(q) = q^{10+51b^2/2+57b/2} \left\langle q^{-50b}; q^{50} \right\rangle_\infty \left\langle -q^{25-15b}; q^{75} \right\rangle_\infty.$$

Here also $F_2(q) = 0$ when b is a positive integer.

Finally, define

$$q^{5+2b} \left\langle q^{2b}; q^5 \right\rangle_\infty \left\langle q^{5-6b}; q^{30} \right\rangle_\infty =: \sum_{n=0}^{\infty} w_n q^n,$$

$$F_3(q) := \sum_{\substack{n=0 \\ n \equiv 3b^2+b \pmod 5}}^{\infty} w_n q^n.$$

Once more employing (1.4) and (1.5),

$$\left\langle q^{2b};q^5\right\rangle_\infty \left\langle q^{5-6b};q^{30}\right\rangle_\infty = \sum_{m,n=-\infty}^{\infty} (-1)^{n+m} q^{15n^2+10n+6bn+2bm+5m(n-1)/2}$$

This time, to get the terms that are in $F_3(q)$, it is necessary and sufficient that $6bn+2bm \equiv 3b^2-b \mod 5$, or that $6n+2m \equiv 3b-1 \mod 5$ and thus $6n-3m \equiv 3b-1 \mod 5$ also. Set $6n+2m = 3b-1+5r$ and $6n-3m = 3b-1+5s$. From the first of these equations it can be seen that r has the form $r = 2j+1+b$, and from the second equation that s has the form $s = 3k+2$ where j and k are integers. Thus $m = r-s = 2j-1+b-3k$, $n = 1+b+j+k$, $(-1)^{n+m} = (-1)^j$,

$$15n^2+10n+6bn+2bm+\frac{5m(m-1)}{2} = 30+\frac{51b^2}{2}+\frac{73b}{2}+25j+50bj+25j^2+15bk+\frac{75k^2+125k}{2},$$

so that, by (1.4) and (1.5) once again,

$$F_3(q) = q^{35+51b^2/2+77b/2}\left\langle q^{-50b};q^{50}\right\rangle_\infty \left\langle q^{-25-15b};q^{75}\right\rangle_\infty = -F_2(q),$$

and $F_3(q)=0$ when b is a positive integer.

Since $F_1(q)=F_2(q)=F_3(q)=0$, then $r_{5n+3b^2+b}=0$ for all n as claimed, and the proof is complete. □

Corollary 2.2. *If the sequences* $\{a_n\}$ *and* $\{b_n\}$ *are defined by*

$$\sum_{n=0}^{\infty} a_n q^n := (q^2,q^3;q^5)_\infty (q^3,q^7;q^{10})^3_\infty, \tag{2.10}$$

$$\sum_{n=0}^{\infty} b_n q^n := (q,q^4;q^5)_\infty (q,q^9;q^{10})^3_\infty, \tag{2.11}$$

then $a_{5n+4}=b_{5n+4}=0$.

Proof. These results are respectively, the cases $b=1$ and $b=2$ of Theorem 2.2. □

3. Mod 7

It is believed that the results in this section are new. It contains a total of six individual results, three following from Theorem 3.1, and three from Theorem 3.2.

Theorem 3.1. *For* $b\in\{1,2,3\}$ *define the sequence* $\{r_n\}$ *by*

$$(q^b,q^{7-b};q^7)_\infty (q^{7-2b},q^{7+2b};q^{14})^3_\infty =: \sum_{n=0}^{\infty} r_n q^n. \tag{3.1}$$

Then $r_{7n+4b} = 0$ *for all* n.

Remark: For $b = 1$, "all n" means $n \geq 0$, while for $b = 2$ or $b = 3$ "all n" means $n \geq -1$.

Proof. In the extended quintuple product formula (1.3) replace q with q^7 and set $a = b = c = -q^{2b}$ to get, after some elementary q-product manipulations, that

$$\begin{aligned}(3.2)\quad & \left\langle q^{7+2b}; q^{14}\right\rangle_\infty^3 = \\ & \left\langle q^{21-6b}; q^{42}\right\rangle_\infty \left\{ \left\langle -q^{14}; q^{28}\right\rangle_\infty \left\langle -q^{42}; q^{84}\right\rangle_\infty + q^{14} \left\langle -1; q^{28}\right\rangle_\infty \left\langle -1; q^{84}\right\rangle_\infty \right\} \\ & \quad - \left(q^{7-2b} \left\langle q^{7+6b}; q^{42}\right\rangle_\infty + q^{7+2b} \left\langle q^{7-6b}; q^{42}\right\rangle_\infty \right) \\ & \quad \times \left\{ \left\langle -1; q^{28}\right\rangle_\infty \left\langle -q^{28}; q^{84}\right\rangle_\infty + \left\langle -q^{14}; q^{28}\right\rangle_\infty \left\langle -q^{14}; q^{84}\right\rangle_\infty \right\}.\end{aligned}$$

Next, multiply both sides by $\left\langle q^b; q^7\right\rangle_\infty$ and isolate those terms in the series expansion with the powers of q that are congruent to $4b$ modulo 7.

Define

$$\left\langle q^b; q^7\right\rangle_\infty \left\langle q^{21-6b}; q^{42}\right\rangle_\infty =: \sum_{n=0}^{\infty} u_n q^n, \qquad F_1(q) := \sum_{\substack{n=0 \\ n \equiv 4b \pmod 7}}^{\infty} u_n q^n.$$

By (1.4) and (1.5),

$$\left\langle q^b; q^7\right\rangle_\infty \left\langle q^{21-6b}; q^{42}\right\rangle_\infty = \sum_{m,n=-\infty}^{\infty} (-1)^{n+m} q^{7n(n-1)/2+bn+6bm+21m^2}$$

To get the terms in this series that are in $F_1(q)$, it is necessary and sufficient that $bn + 6bm \equiv 4b \mod 7$, or that $n + 6m \equiv 4 \mod 7$ and thus $n - m \equiv 4 \mod 7$. Set $n + 6m = 4 + 7r$ and $n - m = 4 + 7s$. Then $m = r - s$, $n = 4 + r + 6s$, $(-1)^{n+m} = (-1)^s$,

$$\frac{7n(n-1)}{2} + bn + 6bm + 21m^2 = 42 + 4b + \frac{49r}{2} + 7br + \frac{49r^2}{2} + 147s + 147s^2$$

and, again using (1.4) and (1.5),

$$F_1(q) = q^{42+4b} \left\langle -q^{-7b}; q^{49}\right\rangle_\infty \left\langle 1; q^{294}\right\rangle_\infty = 0.$$

Similarly, define

$$q^{7-2b} \left\langle q^b; q^7\right\rangle_\infty \left\langle q^{7+6b}; q^{42}\right\rangle_\infty =: \sum_{n=0}^{\infty} v_n q^n, \qquad F_2(q) := \sum_{\substack{n=0 \\ n \equiv 4b \pmod 7}}^{\infty} v_n q^n.$$

By (1.4) and (1.5),

$$\left\langle q^b;q^7\right\rangle_\infty \left\langle q^{7+6b};q^{42}\right\rangle_\infty = \sum_{m,n=-\infty}^{\infty} (-1)^{n+m} q^{7n(n-1)/2+bn+(14-6b)m+21m^2}$$

This time, to get the terms that are in $F_2(q)$, it is necessary and sufficient that $bn-6bm \equiv 6b \mod 7$, or that $n-6m \equiv 6 \equiv -1 \mod 7$ and thus $n+m \equiv -1 \mod 7$ also. This time, set $n-6m = -1+7r$ and $n+m = -1+7s$. Then $m = s-r$, $n = -1+r+6s$, $(-1)^{n+m} = -(-1)^s$,

$$\frac{7n(n-1)}{2}+bn+(14-6b)m+21m^2 = 7-b-\frac{49r}{2}+7br+\frac{49r^2}{2}-49s+147s^2$$

and

$$F_2(q) = -q^{14-3b}\left\langle -q^{7b};q^{49}\right\rangle_\infty \left\langle q^{98};q^{294}\right\rangle_\infty.$$

Finally, define

$$q^{7+2b}\left\langle q^b;q^7\right\rangle_\infty \left\langle q^{7-6b};q^{42}\right\rangle_\infty =: \sum_{n=0}^{\infty} w_n q^n, \qquad F_3(q) := \sum_{\substack{n=0\\ n\equiv 4b \pmod 7}}^{\infty} w_n q^n.$$

Once again, by (1.4) and (1.5),

$$\left\langle q^b;q^7\right\rangle_\infty \left\langle q^{7-6b};q^{42}\right\rangle_\infty = \sum_{m,n=-\infty}^{\infty} (-1)^{n+m} q^{7n(n-1)/2+bn+(14+6b)m+21m^2}$$

This time, to get the terms that are in $F_3(q)$, it is necessary and sufficient that $bn+6bm \equiv 2b \mod 7$, or that $n-m \equiv 2 \mod 7$ and thus $n+6m \equiv 2 \mod 7$ also. This time, set $n-m = 2+7s$ and $n+6m = 2+7r$. Then $m = r-s$, $n = 2+r+6s$, $(-1)^{n+m} = (-1)^s$,

$$\frac{7n(n-1)}{2}+bn+(14+6b)m+21m^2 = 7+2b+\frac{49r}{2}+7br+\frac{49r^2}{2}+49s+147s^2$$

and

$$\begin{aligned} F_3(q) &= q^{14+4b}\left\langle -q^{-7b};q^{49}\right\rangle_\infty \left\langle q^{98};q^{294}\right\rangle_\infty \\ &= q^{14-3b}\left\langle -q^{7b};q^{49}\right\rangle_\infty \left\langle q^{98};q^{294}\right\rangle_\infty = -F_2(q). \end{aligned}$$

Since $F_1(q)=0$ and $F_2(q)+F_3(q)=0$, then $r_{7n+4b}=0$ for all n as claimed, and the proof is complete. $\square$

Corollary 3.1. *If the sequences* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are defined by*

$$\sum_{n=0}^{\infty} a_n q^n := (q, q^6; q^7)_\infty (q^5, q^9; q^{14})_\infty^3,$$

$$\sum_{n=0}^{\infty} b_n q^n := (q^2, q^5; q^7)_\infty (q^3, q^{11}; q^{14})_\infty^3,$$

$$\sum_{n=0}^{\infty} c_n q^n := (q^3, q^4; q^7)_\infty (q, q^{13}; q^{14})_\infty^3,$$

then $a_{7n+4} = b_{7n+1} = c_{7n+5} = 0$.

Proof. Set, in turn, $b = 1, 2, 3$ in Theorem 3.1. □

Theorem 3.2. *For* $b \in \{1, 2, 3\}$ *define the sequence* $\{r_n\}$ *by*

$$(q^{2b}, q^{7-2b}; q^7)_\infty^3 (q^{7-2b}, q^{7+2b}; q^{14})_\infty =: \sum_{n=0}^{\infty} r_n q^n. \tag{3.3}$$

Then $r_{7n+5b^2+3b} = 0$ *for all* n.

Proof. In (1.3), replace q with $q^{7/2}$ and set $a = b = c = -q^{7/2-2b}$ to get that

$$\begin{aligned}\left\langle q^{2b}; q^7 \right\rangle_\infty^3 = &\\ \left\langle q^{6b}; q^{21} \right\rangle_\infty &\Big\{ \left\langle -q^7; q^{14} \right\rangle_\infty \left\langle -q^{21}; q^{42} \right\rangle_\infty + q^7 \left\langle -1; q^{14} \right\rangle_\infty \left\langle -1; q^{42} \right\rangle_\infty \Big\} \\ &+ \left(q^{14-4b} \left\langle q^{-14+6b}; q^{21} \right\rangle_\infty - q^{7-2b} \left\langle q^{-7+6b}; q^{21} \right\rangle_\infty \right) \\ &\times \Big\{ \left\langle -1; q^{14} \right\rangle_\infty \left\langle -q^{14}; q^{42} \right\rangle_\infty + \left\langle -q^7; q^{14} \right\rangle_\infty \left\langle -q^7; q^{42} \right\rangle_\infty \Big\}.\end{aligned} \tag{3.4}$$

The rest of the proof now follows what is becoming a familiar pattern. Multiply both sides by $\left\langle q^{7+2b}; q^{14} \right\rangle_\infty$ and isolate those terms in the series expansion with the powers of q that are congruent to $5b^2 + 3b$ modulo 7.

Define

$$\left\langle q^{7+2b}; q^{14} \right\rangle_\infty \left\langle q^{6b}; q^{21} \right\rangle_\infty =: \sum_{n=0}^{\infty} u_n q^n, \qquad F_1(q) := \sum_{\substack{n=0 \\ n \equiv 5b^2+3b \pmod 7}}^{\infty} u_n q^n.$$

Once again applying (1.4) and (1.5),

$$\left\langle q^{7+2b}; q^{14} \right\rangle_\infty \left\langle q^{6b}; q^{21} \right\rangle_\infty = \sum_{m,n=-\infty}^{\infty} (-1)^{n+m} q^{7n^2+2bn+6bm+21m(m-1)/2}.$$

For terms in this series that are in $F_1(q)$, it is necessary and sufficient that $2bn + 6bm \equiv 5b^2 + 3b \mod 7$, or that $2n + 6m \equiv 5b + 3 \mod 7$ and thus $2n - m \equiv 5b + 3 \mod 7$ also. Set $2n + 6m = 5b + 3 + 7r$ and $2n - m = 5b + 3 + 7s$, and from the first of these it can be seen that r has to have the form $r = 2k - b + 1$. Hence $m = 1 - b + 2k - s$, $n = 2 + 2b + k + 3s$, $(-1)^{n+m} = (-1)^{b+1+k}$,

$$7n^2 + 2bn + 6bm + 21\frac{m(m-1)}{2} = 28 + \frac{(73b+111)b}{2} + 49k + 49k^2 + 105bs + \frac{147(s^2+s)}{2},$$

so that, by (1.4) and (1.5) once again,

$$F_1(q) = (-1)^{b+1} q^{28+(73b+111)b/2} \left\langle 1; q^{98} \right\rangle_\infty \left\langle -q^{-105b}; q^{147} \right\rangle_\infty = 0.$$

In similar fashion, define

$$q^{14-4b} \left\langle q^{-14+6b}; q^{21} \right\rangle_\infty \left\langle q^{7+2b}; q^{14} \right\rangle_\infty =: \sum_{n=0}^{\infty} v_n q^n,$$

$$F_2(q) := \sum_{\substack{n=0 \\ n \equiv 5b^2+3b \pmod 7}}^{\infty} v_n q^n.$$

By (1.4) and (1.5),

$$\left\langle q^{-14+6b}; q^{21} \right\rangle_\infty \left\langle q^{7+2b}; q^{14} \right\rangle_\infty = \sum_{m,n=-\infty}^{\infty} (-1)^{n+m} q^{7n^2+2bn+(-14+6b)m+21m(m-1)/2}.$$

Upon noting the factor q^{14-4b}, to get the terms that are in $F_2(q)$, it is necessary and sufficient that $2bn + 6bm \equiv 5b^2 + 7b \equiv 5b^2 \mod 7$, or that $2n + 6m \equiv 5b \mod 7$ and thus $2n - m \equiv 5b \mod 7$ also. Set $2n + 6m = 5b + 7r$ and $2n - m = 5b + 7s$, and from the first of these it can be seen that r has to have the form $r = 2k - b$. Hence $m = 2k - b - s$, $n = 2b + k + 3s$, $(-1)^{n+m} = (-1)^{b+k}$,

$$7n^2 + 2bn + (-14+6b)m + 21\frac{m(m-1)}{2} = \frac{(73b+49)b}{2} - 49k + 49k^2 + 105bs + \frac{(147s^2+49s)}{2},$$

so that, by (1.4) and (1.5) once again,

$$F_2(q) = (-1)^b q^{14+(73b+41)b/2} \left\langle 1; q^{98} \right\rangle_\infty \left\langle -q^{98-105b}; q^{147} \right\rangle_\infty = 0.$$

Lastly, define

$$-q^{7-2b}\left\langle q^{-7+6b};q^{21}\right\rangle_{\infty}\left\langle q^{7+2b};q^{14}\right\rangle_{\infty}=:\sum_{n=0}^{\infty}w_nq^n,$$

$$F_3(q):=\sum_{\substack{n=0\\ n\equiv 6b^2+b \pmod 7}}^{\infty}w_nq^n.$$

Once again, by (1.4) and (1.5),

$$\left\langle q^{-7+6b};q^{21}\right\rangle_{\infty}\left\langle q^{7+2b};q^{14}\right\rangle_{\infty}$$
$$=\sum_{m,n=-\infty}^{\infty}(-1)^{n+m}q^{7n^2+2bn+(-14+6b)m+21m(m-1)/2}.$$

Upon taking account of the factor q^{7-2b}, to get the terms that are in $F_3(q)$, it is necessary and sufficient that $2bn+6bm\equiv 5b^2+5b \mod 7$, or that $2n+6m\equiv 5b+5 \mod 7$ and thus $2n-m\equiv 5b+5 \mod 7$ also. Set $2n+6m=5b+5+7r$ and $2n-m=5b+5+7s$, and from the first of these it can be seen that r has to have the form $r=2k-b-1$. Hence $m=2k-b-1-s$, $n=2b+k+3s+2$, $(-1)^{n+m}=(-1)^{b+k+1}$,

$$7n^2+2bn+(-14+6b)m+21\frac{m(m-1)}{2}$$
$$=56+\frac{(73b+185)b}{2}-49k+49k^2+105bs+\frac{(147s^2+245s)}{2},$$

so that, by (1.4) and (1.5) once again,

$$F_3(q)=(-1)^{b+1}q^{63+(73b+181)b/2}\left\langle 1;q^{98}\right\rangle_{\infty}\left\langle -q^{-98-105b};q^{147}\right\rangle_{\infty}=0.$$

Since $F_1(q)=F_2(q)=F_3(q)=0$, then $r_{7n+5b^2+3b}=0$ for all n as claimed, and the proof is complete. □

Corollary 3.2. *If the sequences* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are defined by*

$$\sum_{n=0}^{\infty}a_nq^n:=(q^2,q^5;q^7)_{\infty}^3(q^5,q^9;q^{14})_{\infty},$$
$$\sum_{n=0}^{\infty}b_nq^n:=(q^3,q^4;q^7)_{\infty}^3(q^3,q^{11};q^{14})_{\infty},$$
$$\sum_{n=0}^{\infty}c_nq^n:=(q,q^6;q^7)_{\infty}^3(q,q^{13};q^{14})_{\infty},$$

then $a_{7n+1}=b_{7n+5}=c_{7n+5}=0$.

Proof. This follows directly from Theorem 3.2, upon setting b, in turn, equal to 1,2,3. □

4. Mod 11

It is believed that the results in this section are also new. It contains a total of thirty individual results, fifteen following from Theorem 4.1, and fifteen from Theorem 4.2.

Remark: When the product $(q^{11-2b};q^{11})_\infty$ is expanded in powers of q, there will be negative powers for $b>5$, and thus the summation variable (n below) will start at a negative integer. Here, and later in the paper where similar situations occur, we make the lower limit of summation the most negative exponent that occurs in the series expansion of the infinite product with the largest value of b in its stated range. This lower limit will then be sufficient for all b.

Theorem 4.1. *For $b\in\{1,2,\dots,9,10\}$ define the sequence $\{r_n\}$ by*

$$(q^{2b},q^{11-2b};q^{11})_\infty(q^{11-b},q^{11+b};q^{22})_\infty^3=:\sum_{n=-9}^{\infty}r_nq^n. \tag{4.1}$$

(i) For all n, $r_{11n+b}=0$.
(ii) In addition, if $b\in\{1,3,5,7,9\}$, then $r_{11n+5b^2+b}=0$ for all n.

Proof. The proof is similar to that of Theorem 3.1. In (1.3) replace q with q^{11} and set $a=b=c=-q^b$ to get, after some elementary q-product manipulations, that

$$\begin{aligned}&\left\langle q^{11+b};q^{22}\right\rangle_\infty^3= \\ &\left\langle q^{33+3b};q^{66}\right\rangle_\infty\left\{\left\langle -q^{22};q^{44}\right\rangle_\infty\left\langle -q^{66};q^{132}\right\rangle_\infty+q^{22}\left\langle -1;q^{44}\right\rangle_\infty\left\langle -1;q^{132}\right\rangle_\infty\right\}\\ &\qquad-\left(q^{11-b}\left\langle q^{11+3b};q^{66}\right\rangle_\infty+q^{11+b}\left\langle q^{11-3b};q^{66}\right\rangle_\infty\right)\\ &\qquad\times\left\{\left\langle -1;q^{44}\right\rangle_\infty\left\langle -q^{44};q^{132}\right\rangle_\infty+\left\langle -q^{22};q^{44}\right\rangle_\infty\left\langle -q^{22};q^{132}\right\rangle_\infty\right\}.\end{aligned} \tag{4.2}$$

(i) Multiply both sides of (4.2) by $\left\langle q^{2b};q^{11}\right\rangle_\infty$ and isolate those terms in the series expansion with the powers of q that are congruent to b modulo 11.

Define

$$\left\langle q^{2b};q^{11}\right\rangle_\infty\left\langle q^{33+3b};q^{66}\right\rangle_\infty=:\sum_{n=0}^{\infty}u_nq^n,\qquad F_1(q):=\sum_{\substack{n=0\\ n\equiv b \pmod{11}}}^{\infty}u_nq^n.$$

Once again applying (1.4) and (1.5),

$$\left\langle q^{2b};q^{11}\right\rangle_\infty\left\langle q^{33-3b};q^{66}\right\rangle_\infty=\sum_{m,n=-\infty}^{\infty}(-1)^{n+m}q^{33n^2+3bn+2bm+11m(m-1)/2}.$$

To get the terms in this series that are in $F_1(q)$, it is necessary and sufficient that $3bn+2bm\equiv b \mod 11$, or that $3n+2m\equiv 1 \mod 11$ and thus $-8n+$

$2m \equiv 1 \mod 11$ also. Set $3n+2m = 1+11r$ and $-8n+2m = 1+11s$, from which it can be seen that s is odd, say $s = 2k+1$. Hence $n = r-2k-1$ and $m = 4r+3k+2$, $(-1)^{n+m} = (-1)^{r+k+1}$,

$$33n^2+3bn+2bm+\frac{11m(m-1)}{2} = 44+b+\frac{363(k^2+k)}{2}+11br+121r^2,$$

so that, by (1.4) and (1.5) once again,

$$F_1(q) = -q^{44+b}\left\langle 1;q^{363}\right\rangle_\infty \left\langle q^{11b+121};q^{242}\right\rangle_\infty = 0.$$

Similarly, define

$$q^{11-b}\left\langle q^{2b};q^{11}\right\rangle_\infty \left\langle q^{11+3b};q^{66}\right\rangle_\infty =: \sum_{n=0}^{\infty} v_n q^n,$$

$$F_2(q) := \sum_{\substack{n=0\\ n\equiv b \pmod{11}}}^{\infty} v_n q^n.$$

By (1.4) and (1.5),

$$\left\langle q^{2b};q^{11}\right\rangle_\infty \left\langle q^{11+3b};q^{66}\right\rangle_\infty = \sum_{m,n=-\infty}^{\infty} (-1)^{n+m} q^{33n^2+(-22+3b)n+2bm+11m(m-1)/2}$$

Upon taking into consideration the factor q^{11-b}, to get the terms in this series that are in $F_2(q)$, it is necessary and sufficient that $3bn+2bm \equiv 2b \mod 11$, or that $3n+2m \equiv 2 \mod 11$ and thus $-8n+2m \equiv 2 \mod 11$ also. Set $3n+2m = 2+11r$ and $-8n+2m = 2+11s$, from which it can be seen that s is even, say $s = 2k$. Hence $n = r-2k$ and $m = 4r+3k+1$, $(-1)^{n+m} = (-1)^{r+k+1}$,

$$33n^2+(-22+3b)n+2bm+\frac{11m(m-1)}{2} = 2b+\frac{(363k+121)k}{2}+11br+121r^2,$$

so that, by (1.4) and (1.5) once again,

$$F_2(q) = -q^{11+b}\left\langle q^{121};q^{363}\right\rangle_\infty \left\langle q^{121+11b};q^{242}\right\rangle_\infty.$$

Finally, define

$$q^{11+b}\left\langle q^{2b};q^{11}\right\rangle_\infty \left\langle q^{11-3b};q^{66}\right\rangle_\infty =: \sum_{n=0}^{\infty} w_n q^n,$$

$$F_3(q) := \sum_{\substack{n=0\\ n\equiv b \pmod{11}}}^{\infty} w_n q^n.$$

By (1.4) and (1.5),

$$\left\langle q^{2b};q^{11}\right\rangle_\infty \left\langle q^{11-3b};q^{66}\right\rangle_\infty = \sum_{m,n=-\infty}^{\infty} (-1)^{n+m} q^{33n^2+(22+3b)n+2bm+11m(m-1)/2}$$

Upon taking into consideration the factor q^{11+b}, to get the terms in this series that are in $F_3(q)$, it is necessary and sufficient that $3bn+2bm \equiv 0 \mod 11$, or that $3n+2m \equiv 0 \mod 11$ and thus $-8n+2m \equiv 0 \mod 11$ also. Set $3n+2m = 11r$ and $-8n+2m = 11s$, from which it can be seen that s is even, say $s=2k$. Hence $n = r-2k$ and $m = 4r+3k$, $(-1)^{n+m} = (-1)^{r+k}$,

$$33n^2+(22+3b)n+2bm+\frac{11m(m-1)}{2} = \frac{(363k-121)k}{2}+11br+121r^2,$$

so that, by (1.4) and (1.5) once again,

$$F_3(q) = q^{11+b}\left\langle q^{121};q^{363}\right\rangle_\infty \left\langle q^{121+11b};q^{242}\right\rangle_\infty = -F_2(q).$$

Since $F_1(q)=0$ and $F_2(q)+F_3(q)=0$, then $r_{11n+b}=0$ for all n as claimed, and the proof of (i) is complete.

For (ii), the analysis is very similar, except the arithmetic progression examined is $11n+5b^2+b$, instead of $11n+b$. Upon employing the same notation, it is found that the conditions m and n need to satisfy to obtain $F_1(q)$, $F_2(q)$ and $F_3(q)$, are as shown in the following table:

$$\begin{aligned} &F_1(q): && 2m+3n \equiv 5b+1 \pmod{11}\\ &F_2(q): && 2m+3n \equiv 5b+2 \pmod{11}\\ &F_3(q): && 2m+3n \equiv 5b \pmod{11}\end{aligned}$$

These congruence conditions lead in turn to the following expressions:

$$\begin{aligned} F_1(q) &= (-1)^{1+b}q^{44+133b+126b^2}\left\langle q^{-264b};q^{363}\right\rangle_\infty \left\langle q^{121(1-b)};q^{242}\right\rangle_\infty,\\ F_2(q) &= (-1)^{1+b}q^{11+45b+126b^2}\left\langle q^{121-264b};q^{363}\right\rangle_\infty \left\langle q^{121(1-b)};q^{242}\right\rangle_\infty,\\ F_3(q) &= (-1)^{b}q^{11-43b+126b^2}\left\langle q^{121-264b};q^{363}\right\rangle_\infty \left\langle q^{121(1-b)};q^{242}\right\rangle_\infty.\end{aligned}$$

It can be seen that $F_1(q)=F_2(q)=F_3(q)=0$ when b is an odd positive integer. Thus $r_{11n+5b^2+b}=0$ for all n as claimed, and the proof of (ii) is also complete. □

Corollary 4.1. *If the sequences* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$, $\{i_n\}$ *and* $\{j_n\}$ *are defined by*

$$\sum_{n=0}^{\infty} a_n q^n := (q^2,q^9;q^{11})_\infty (q^{10},q^{12};q^{22})^3_\infty,$$

$$\sum_{n=0}^{\infty} b_n q^n := (q^4, q^7; q^{11})_\infty (q^9, q^{13}; q^{22})_\infty^3,$$
$$\sum_{n=0}^{\infty} c_n q^n := (q^5, q^6; q^{11})_\infty (q^8, q^{14}; q^{22})_\infty^3,$$
$$\sum_{n=0}^{\infty} d_n q^n := (q^3, q^8; q^{11})_\infty (q^7, q^{15}; q^{22})_\infty^3,$$
$$\sum_{n=0}^{\infty} e_n q^n := (q, q^{10}; q^{11})_\infty (q^6, q^{16}; q^{22})_\infty^3,$$
$$\sum_{n=0}^{\infty} f_n q^n := (q, q^{10}; q^{11})_\infty (q^5, q^{17}; q^{22})_\infty^3,$$
$$\sum_{n=0}^{\infty} g_n q^n := (q^3, q^8; q^{11})_\infty (q^4, q^{18}; q^{22})_\infty^3,$$
$$\sum_{n=0}^{\infty} h_n q^n := (q^5, q^6; q^{11})_\infty (q^3, q^{19}; q^{22})_\infty^3,$$
$$\sum_{n=0}^{\infty} i_n q^n := (q^4, q^7; q^{11})_\infty (q^2, q^{20}; q^{22})_\infty^3,$$
$$\sum_{n=0}^{\infty} j_n q^n := (q^2, q^9; q^{11})_\infty (q, q^{21}; q^{22})_\infty^3,$$

then $a_{11n+1} = a_{11n+6} = b_{11n+2} = c_{11n+3} = c_{11n+4} = d_{11n+4} = e_{11n+5} = e_{11n+9} = f_{11n+7} = g_{11n+2} = g_{11n+10} = h_{11n+2} = i_{11n+3} = i_{11n+5} = j_{11n+8} = 0$.

Proof. These results follow from Theorem 4.1, upon letting b assume, in turn, the values $1, \dots 10$. Note that for $b \geq 6$, it is necessary to slightly modify the infinite product at (4.1) (by performing elementary q-product manipulations and then multiplying both sides by some power of q) to make them have the form of the infinite products in the corollary. This will cause a shift in the arithmetic progression predicted by the theorem. □

Theorem 4.2. *For* $b \in \{1, 2, \dots, 9, 10\}$ *define the sequence* $\{r_n\}$ *by*

$$(q^{8b}, q^{11-8b}; q^{11})_\infty^3 (q^{11-b}, q^{11+b}; q^{22})_\infty =: \sum_{n=-756}^{\infty} r_n q^n. \tag{4.3}$$

(i) For all n, $r_{11n+6b^2+b} = 0$.
(ii) In addition, if $b \in \{1, 3, 5, 7, 9\}$, *then* $r_{11n+4b^2+b} = 0$ *for all* n.

Remark: In (i), "for all n" means all n for which $11n + 6b^2 + b \geq 0$, while in (ii) it means all n for which $11n + 4b^2 + b \geq 0$

Proof. We first prove the claim in (i). In (1.3), replace q with $q^{11/2}$ and set $a = b = c = -q^{11/2-8b}$ to get that

$$(4.4)\quad \left\langle q^{8b}; q^{11}\right\rangle_\infty^3 =$$
$$\left\langle q^{24b}; q^{33}\right\rangle_\infty \left\{ \left\langle -q^{11}; q^{22}\right\rangle_\infty \left\langle -q^{33}; q^{66}\right\rangle_\infty + q^{11}\left\langle -1; q^{22}\right\rangle_\infty \left\langle -1; q^{66}\right\rangle_\infty \right\}$$
$$+ \left(q^{22-16b}\left\langle q^{-22+24b}; q^{33}\right\rangle_\infty - q^{11-8b}\left\langle q^{-11+24b}; q^{33}\right\rangle_\infty \right)$$
$$\times \left\{ \left\langle -1; q^{22}\right\rangle_\infty \left\langle -q^{22}; q^{66}\right\rangle_\infty + \left\langle -q^{11}; q^{22}\right\rangle_\infty \left\langle -q^{11}; q^{66}\right\rangle_\infty \right\}.$$

The remainder of the proof follows the familiar pattern of previous proofs. Multiply both sides by $\left\langle q^{11+b}; q^{22}\right\rangle_\infty$ and isolate those terms in the series expansion with the powers of q that are congruent to $6b^2 + b$ modulo 11.

Define

$$\left\langle q^{11+b}; q^{22}\right\rangle_\infty \left\langle q^{24b}; q^{33}\right\rangle_\infty =: \sum_{n=0}^{\infty} u_n q^n, \qquad F_1(q) := \sum_{\substack{n=0\\ n\equiv 6b^2+b \pmod{11}}}^{\infty} u_n q^n.$$

Once again applying (1.4) and (1.5),

$$\left\langle q^{11+b}; q^{22}\right\rangle_\infty \left\langle q^{24b}; q^{33}\right\rangle_\infty = \sum_{m,n=-\infty}^{\infty} (-1)^{n+m} q^{11n^2+bn+24bm+33m(m-1)/2}.$$

To get the terms in this series that are in $F_1(q)$, it is necessary and sufficient that $bn + 2bm \equiv 6b^2 + b \mod 11$, or that $n + 2m \equiv 6b + 1 \mod 11$ and thus $-10n + 2m \equiv 6b + 1 \mod 11$ also. Set $n + 2m = 6b + 1 + 11r$ and $-10n + 2m = 6b + 1 + 11s$, from which it can be seen that s is odd, say $s = 2k + 1$. Hence $n = r - 2k - 1$ and $m = 3b + 5r + k + 1$. However, this results in the double series for $F_1(q)$ having $121kr$ in the exponent of q, and a further change of variable, $k \to k - r - b$ is necessary to remove these "cross terms". Thus finally $m = 1 + 2b + k + 4r$, $n = -1 + 2b - 2k + 3r$, $(-1)^{n+m} = (-1)^{r+k}$,

$$11n^2 + bn + 24bm + \frac{33m(m-1)}{2}$$
$$= 11 + 12b + 160b^2 + \frac{121(k^2+k)}{2} + 495br + 363r^2,$$

so that, by (1.4) and (1.5) once again,

$$F_1(q) = q^{11+12b+160b^2}\left\langle 1; q^{121}\right\rangle_\infty \left\langle q^{495b+363}; q^{726}\right\rangle_\infty = 0.$$

Similarly, define

$$q^{22-16b}\left\langle q^{11+b};q^{22}\right\rangle_\infty\left\langle q^{-22+24b};q^{33}\right\rangle_\infty=:\sum_{n=0}^{\infty}v_nq^n,$$

$$F_2(q):=\sum_{\substack{n=0\\ n\equiv 6b^2+b \pmod{11}}}^{\infty}v_nq^n.$$

By (1.4) and (1.5),

$$\left\langle q^{11+b};q^{22}\right\rangle_\infty\left\langle q^{-22+24b};q^{33}\right\rangle_\infty \\ =\sum_{m,n=-\infty}^{\infty}(-1)^{n+m}q^{11n^2+bn+(-22+24b)m+33m(n-1)/2}$$

Upon taking into consideration the factor q^{22-16b}, to get the terms that are in $F_2(q)$, it is necessary and sufficient that $bn+2bm\equiv 6b^2+6b \mod 11$, or that $n+2m\equiv 6b+6 \mod 11$ and thus $-10n+2m\equiv 6b+6 \mod 11$ also. This time, set $n+2m=6b+6+11r$ and $-10n+2m=6b+6+11s$. From the last equation it can be seen that s is even, say $s=2k$. Hence $n=r-2k$ and $m=3b+5r+k+3$. However, as above, this results in the double series for $F_2(q)$ having $121kr$ in the exponent of q, and the same further change of variable, $k\to k-r-b$, is necessary. This gives finally that $m=3+2b+k+4r$, $n=2b-2k+3r$, $(-1)^{n+m}=(-1)^{r+k+1}$,

$$11n^2+bn+(-22+24b)m+\frac{33m(m-1)}{2} \\ =33+193b+160b^2+\frac{121(k^2+k)}{2}+242r+495br+363r^2,$$

so that, by (1.4) and (1.5) once again,

$$F_2(q)=-q^{55+177b+160b^2}\left\langle 1;q^{121}\right\rangle_\infty\left\langle q^{121-495b};q^{726}\right\rangle_\infty=0.$$

Finally, define

$$-q^{11-8b}\left\langle q^{11+b};q^{22}\right\rangle_\infty\left\langle q^{-11+24b};q^{33}\right\rangle_\infty=:\sum_{n=0}^{\infty}w_nq^n,$$

$$F_3(q):=\sum_{\substack{n=0\\ n\equiv 6b^2+b \pmod{11}}}^{\infty}w_nq^n.$$

Once again, by (1.4) and (1.5),

$$\left\langle q^{11+b};q^{22}\right\rangle_\infty \left\langle q^{-11+24b};q^{33}\right\rangle_\infty = \sum_{m,n=-\infty}^{\infty}(-1)^{n+m}q^{11n^2+bn+(-11+24b)m+33m(n-1)/2}$$

This time, to get the terms that are in $F_3(q)$, it is necessary and sufficient (after taking account of the q^{11-8b} factor) that $bn+2bm \equiv 6b^2+9b \mod 11$, or that $n+2m \equiv 6b+9 \mod 11$ and thus $-10n+2m \equiv 6b+9 \mod 11$ also. This time, set $n+2m = 6b+9+11r$ and $-10n+2m = 6b+9+11s$. From the last equation it can be seen that s is odd, say $s = 2k+1$. Hence $n = r-2k-1$ and $m = 3b+5r+k+5$. However, as above, this results in the double series for $F_3(q)$ having $121kr$ in the exponent of q, and a further change of variable, this time $k \to k-r-b-1$, is necessary. This gives finally that $m = 4+2b+k+4r$, $n = 1+2b-2k+3r$, $(-1)^{n+m} = (-1)^{r+k+1}$,

$$11n^2+bn+(-11+24b)m+\frac{33m(m-1)}{2} = 165+350b+160b^2+\frac{121(k^2+k)}{2}+484r+495br+363r^2,$$

so that, by (1.4) and (1.5) once again,

$$F_3(q) = q^{176+342b+160b^2}\left\langle 1;q^{121}\right\rangle_\infty \left\langle q^{-121-495b};q^{726}\right\rangle_\infty = 0.$$

Since $F_1(q) = F_2(q) = F_3(q) = 0$, then $r_{11n+6b^2+b} = 0$ for all n as claimed, and the proof of (i) is complete.

For (ii), the analysis is very similar, except that $6b^2+b$ is replaced everywhere with $4b^2+b$. Upon employing the same notation, it is found that

$$F_1(q) = q^{11+56b+114b^2}\left\langle q^{-88b};q^{121}\right\rangle_\infty \left\langle q^{363-363b};q^{726}\right\rangle_\infty,$$

and the second infinite product is equal to 0 for b odd.

Likewise, it is found that

$$F_2(q) = -q^{55+177b+160b^2}\left\langle q^{-88b};q^{121}\right\rangle_\infty \left\langle q^{121-363b};q^{726}\right\rangle_\infty,$$
$$F_3(q) = -q^{55-65b+160b^2}\left\langle q^{-88b};q^{121}\right\rangle_\infty \left\langle q^{121+363b};q^{726}\right\rangle_\infty.$$

A simple induction argument shows that $F_2(q)+F_3(q) = 0$ when b is an odd positive integer. Thus $r_{11n+4b^2+b} = 0$ for all n as claimed, and the proof of (ii) is also complete. □

Corollary 4.2. *Let the sequences* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$, $\{i_n\}$ *and* $\{j_n\}$ *are defined by*

$$\sum_{n=0}^{\infty} a_n q^n := (q^3,q^8;q^{11})_\infty^3 (q^{10},q^{12};q^{22})_\infty,$$

$$\sum_{n=0}^{\infty} b_n q^n := (q^5, q^6; q^{11})_\infty^3 (q^9, q^{13}; q^{22})_\infty,$$
$$\sum_{n=0}^{\infty} c_n q^n := (q^2, q^9; q^{11})_\infty^3 (q^8, q^{14}; q^{22})_\infty,$$
$$\sum_{n=0}^{\infty} d_n q^n := (q, q^{10}; q^{11})_\infty^3 (q^7, q^{15}; q^{22})_\infty,$$
$$\sum_{n=0}^{\infty} e_n q^n := (q^4, q^7; q^{11})_\infty^3 (q^6, q^{16}; q^{22})_\infty,$$
$$\sum_{n=0}^{\infty} f_n q^n := (q^4, q^7; q^{11})_\infty^3 (q^5, q^{17}; q^{22})_\infty,$$
$$\sum_{n=0}^{\infty} g_n q^n := (q, q^{10}; q^{11})_\infty^3 (q^4, q^{18}; q^{22})_\infty,$$
$$\sum_{n=0}^{\infty} h_n q^n := (q^2, q^9; q^{11})_\infty^3 (q^3, q^{19}; q^{22})_\infty,$$
$$\sum_{n=0}^{\infty} i_n q^n := (q^5, q^6; q^{11})_\infty^3 (q^2, q^{20}; q^{22})_\infty,$$
$$\sum_{n=0}^{\infty} j_n q^n := (q^3, q^8; q^{11})_\infty^3 (q, q^{21}; q^{22})_\infty,$$

then $a_{11n+5} = a_{11n+7} = b_{11n+8} = c_{11n+3} = c_{11n+7} = d_{11n+6} = e_{11n+3} = e_{11n+9} = f_{11n+6} = g_{11n+8} = g_{11n+9} = h_{11n+10} = i_{11n+1} = i_{11n+9} = j_{11n+2} = 0$.

Proof. These results follow directly from Theorem 4.2, upon letting b assume, in turn, the values $1, \dots 10$. Note that for $b \geq 2$, it is necessary to slightly modify the infinite product at (4.3) (by performing elementary q-product manipulations and then multiplying both sides by some power of q) to make it have the form of the infinite product in each case after the first in the corollary. This in turn leads to a shift in the predicted arithmetic progression. □

5. Infinite Products with Negative Signs

The following results may be proved by using the same methods which were used above. Some particular instances of these results may also be found in the papers of Baruah and Kaur [1], Hirschhorn [4] and Tang [6] mentioned in the introduction.

Theorem 5.1. *Let $b \in \{1, 2, 3, 4\}$.*
(i) Define the sequence $\{r_n\}$ by

$$(-q^{2b}, -q^{5-2b}; q^5)^3_\infty (q^{5-b}, q^{5+b}; q^{10})_\infty =: \sum_{n=-9}^{\infty} r_n q^n. \tag{5.1}$$

Then $r_{5n+3b^2+3b} = 0$ for all n. In addition, if b is even, then $r_{5n+2b^2+3b} = 0$ for all n.
(ii) Define the sequence $\{r_n\}$ by

$$(q^{2b}, q^{5-2b}; q^5)^3_\infty (-q^{5-b}, -q^{5+b}; q^{10})_\infty =: \sum_{n=-9}^{\infty} r_n q^n. \tag{5.2}$$

Then $r_{5n+3b^2+3b} = 0$ for all n.
(iii) Define the sequence $\{r_n\}$ by

$$(-q^{b}, -q^{5-b}; q^5)_\infty (q^{5-b}, q^{5+b}; q^{10})^3_\infty =: \sum_{n=0}^{\infty} r_n q^n. \tag{5.3}$$

Then $r_{5n+3b} = 0$ for all n. In addition, if b is even, then $r_{5n+2b^2+3b} = 0$ for all n.
(iv) Define the sequence $\{r_n\}$ by

$$(q^{b}, q^{5-b}; q^5)_\infty (-q^{5-b}, -q^{5+b}; q^{10})^3_\infty =: \sum_{n=0}^{\infty} r_n q^n. \tag{5.4}$$

Then $r_{5n+3b} = 0$ for all n.

There are four similar families of results for arithmetic progressions modulo 7.

Theorem 5.2. *Let $b \in \{1, 2, 3, 4, 5, 6\}$.*
(i) Define the sequence $\{r_n\}$ by

$$(-q^{b}, -q^{7-b}; q^7)^3_\infty (q^{7-b}, q^{7+b}; q^{14})_\infty =: \sum_{n=0}^{\infty} r_n q^n. \tag{5.5}$$

Then $r_{7n+5b} = 0$ for all n. In addition, if b is even, then $r_{7n+3b^2+5b} = 0$ for all n.
(ii) Define the sequence $\{r_n\}$ by

$$(q^{b}, q^{7-b}; q^7)^3_\infty (-q^{7-b}, -q^{7+b}; q^{14})_\infty =: \sum_{n=0}^{\infty} r_n q^n. \tag{5.6}$$

Then $r_{7n+5b} = 0$ for all n.
(iii) Define the sequence $\{r_n\}$ by

$$(-q^{3b}, -q^{7-3b}; q^7)_\infty (q^{7-b}, q^{7+b}; q^{14})^3_\infty =: \sum_{n=-15}^{\infty} r_n q^n. \tag{5.7}$$

Then $r_{7n+5b}=0$ *for all* n*. In addition, if* b *is even, then* $r_{7n+2b^2+5b}=0$ *for all* n*.*
(iv) Define the sequence $\{r_n\}$ *by*

$$(q^{4b},q^{7-4b};q^7)_\infty(-q^{7-b},-q^{7+b};q^{14})^3_\infty=:\sum_{n=-30}^{\infty}r_nq^n. \tag{5.8}$$

Then $r_{7n+3b^2+b}=0$ *for all* n*.*

Finally, here are the results for arithmetic progressions modulo 11.

Theorem 5.3. *Let* $b\in\{1,2,3,4,5,6,7,8,9,10\}$*.*
(i) Define the sequence $\{r_n\}$ *by*

$$(-q^{3b},-q^{11-3b};q^{11})^3_\infty(q^{11-b},q^{11+b};q^{22})_\infty=:\sum_{n=-81}^{\infty}r_nq^n. \tag{5.9}$$

Then $r_{11n+6b^2+10b}=0$ *for all* n*.*
(ii) Define the sequence $\{r_n\}$ *by*

$$(q^{3b},q^{11-3b};q^{11})^3_\infty(-q^{11-b},-q^{11+b};q^{22})_\infty=:\sum_{n=-81}^{\infty}r_nq^n. \tag{5.10}$$

Then $r_{11n+8b^2+10b}=0$ *for all* n*.*
(iii) Define the sequence $\{r_n\}$ *by*

$$(-q^{2b},-q^{11-2b};q^{11})_\infty(q^{11-b},q^{11+b};q^{22})^3_\infty=:\sum_{n=-9}^{\infty}r_nq^n. \tag{5.11}$$

Then $r_{11n+5b^2+b}=0$ *for all* n*.*
(iv) Define the sequence $\{r_n\}$ *by*

$$(q^{2b},q^{11-2b};q^{11})_\infty(-q^{11-b},-q^{11+b};q^{22})^3_\infty=:\sum_{n=-9}^{\infty}r_nq^n. \tag{5.12}$$

Then $r_{11n+b}=0$ *for all* n*.*

6. m-DISSECTIONS

By the *m-dissections* of an infinite product $f(q)$ is meant a representation of the form

$$f(q)=f_0(q^m)+qf_1(q^m)+q^2f_2(q^m)+\cdots+q^{m-1}f_{m-1}(q^m),$$

where each $f_i(q^m)$ consists of a finite sum of products/quotients of infinite q-products such that the series expansions contains only powers of q^m.

In a recent paper [7], Tang and Xia give 5-dissections for the infinite products $(-q,-q^4;q^5)^2_\infty(q^4,q^6;q^{10})_\infty$ and $(-q^2,-q^3;q^5)^2_\infty(q^2,q^8;q^{10})_\infty$ (amongst other results). Dou and Xiao in [3] similarly treated the infinite products $(q,q^4;q^5)_\infty(q^3,q^7;q^{10})^2_\infty$ and $(q^2,q^3;q^5)_\infty(q,q^9;q^{10})^2_\infty$.

The methods in the present paper also result in similar dissections of the infinite products considered here. We give just one example (this being the case $b=1$ of (2.1)) as a full treatment of the results would add too much to the length of the present paper (these dissections will be discussed in full in a future paper).

It may be seen from the $b=1$ case of (2.2) that a 5-dissection of $\left\langle q;q^5\right\rangle_\infty^3\left\langle q^3;q^{10}\right\rangle_\infty$ may be produced if the 5-dissections of the products $\left\langle q^3;q^{15}\right\rangle_\infty\left\langle q^3;q^{10}\right\rangle_\infty$, $\left\langle q^{-7};q^{15}\right\rangle_\infty\left\langle q^3;q^{10}\right\rangle_\infty$ and $\left\langle q^{-2};q^{15}\right\rangle_\infty\left\langle q^3;q^{10}\right\rangle_\infty$ can be found.

Firstly, by considering the $b=1$ case of (2.3) and slightly generalizing the discussion following, it can be seen that, for $a\in\{0,1,2,3,4\}$, that all powers of q with exponent congruent to a mod5 may be grouped together by restricting to those m and n such that $2n+3m\equiv a \mod 5$ and thus $-3n+3m\equiv a \mod 5$ also. Set $2n+3m=a+5r$ and $-3n+3m=a+5s$, from which it can be seen that s has the form $s=3k+a$ for k an integer. Hence $n=r-3k-a$ and $m=a+r+2k$, $(-1)^{n+m}=(-1)^k$,

$$\begin{aligned}5n^2+2n+3m+&\frac{15m(m-1)}{2}\\ &=\frac{a\,(25a-13)}{2}+15(4a-1)k+75k^2+\frac{5(2a-1)r+25r^2}{2},\end{aligned}$$

so that, by summing over all $a\in\{0,1,2,3,4\}$ and using (1.4) and (1.5) once again,

$$\begin{aligned}&\left\langle q^3;q^{15}\right\rangle_\infty\left\langle q^3;q^{10}\right\rangle_\infty\\ &\qquad=\sum_{a=0}^{5}q^{a(25a-13)/2}\left\langle q^{75+15(4a-1)};q^{150}\right\rangle_\infty\left\langle -q^{5a+10};q^{25}\right\rangle_\infty,\end{aligned}$$

thus providing the first of the three required 5-dissections.

By applying similar reasoning to the $b=1$ case of, respectively, (2.4) and (2.5), one gets that

$$\begin{aligned}&\left\langle q^{-7};q^{15}\right\rangle_\infty\left\langle q^3;q^{10}\right\rangle_\infty\\ &\qquad=\sum_{a=0}^{5}q^{a(25a-33)/2}\left\langle q^{75+60a-35)};q^{150}\right\rangle_\infty\left\langle -q^{5a};q^{25}\right\rangle_\infty\end{aligned}$$

and

$$\begin{aligned}&\left\langle q^{-2};q^{15}\right\rangle_\infty\left\langle q^3;q^{10}\right\rangle_\infty\\ &\qquad=\sum_{a=0}^{5}q^{a(25a-23)/2}\left\langle q^{75+60a-25)};q^{150}\right\rangle_\infty\left\langle -q^{5a+5};q^{25}\right\rangle_\infty.\end{aligned}$$

As noted above, these three 5-dissections my be inserted into the $b=1$ case of (2.2) to produce a 5-dissection of $\left\langle q;q^5\right\rangle_\infty^3\left\langle q^3;q^{10}\right\rangle_\infty$. Similar arguments may be used to produce dissections of other infinite products in the paper.

7. Concluding Remarks

As Tang remarks, there seems to be no vanishing coefficient results for $t = 13$ or $t = 17$ for the infinite products at (1.2), and the same seems to hold for the other kinds of infinite product examined in the present paper. However, it would be interesting to see if these formats may be modified in any way so as to produce new classes of infinite products with coefficients vanishing in arithmetic progressions.

References

[1] Baruah, N.D., Kaur, M. *Some results on vanishing coefficients in infinite product expansions* Ramanujan J (2019). https://doi.org/10.1007/s11139-019-00172-x, published online 05 September 2019.

[2] Cao, Z. *Integer Matrix Exact Covering Systems and Product Identities for Theta Functions.* Int. Math. Res. Not. no. **19** (2011), 44714514.

[3] Dou, D.Q.J., Xiao, J. *The 5-dissections of two infinite product expansions.* Ramanujan J (2020). https://doi.org/10.1007/s11139-019-00200-w

[4] Hirschhorn, M. D. *Two remarkable q-series expansions.* Ramanujan J. **49**(2019), no. 2, pp 451–463.

[5] Mc Laughlin, J., *A Generalization of Schröter's Formula.* - To appear in Special Issue of the Annals of Combinatorics containing the conference proceedings of the Combinatory Analysis 2018 conference (held to honour George Andrews on the occasion of his 80th birthday).

[6] Tang, D. *Vanishing coefficients in some q-series expansions.* Int. J. Number Theory **15** (2019), no. 4, 763–773.

[7] Tang, D., Xia, E.X.W. *Several q-series related to Ramanujans theta functions.* Ramanujan J (2019). https://doi.org/10.1007/s11139-019-00187-4

Mathematics Department, 25 University Avenue, West Chester University, West Chester, PA 19383

E-mail address: jmclaughl@wcupa.edu