FURTHER RESULTS ON VANISHING COEFFICIENTS IN INFINITE PRODUCTS OF THE FORM $(q^b, q^{p-b}; q^p)_\infty^3 (q^{jb}, q^{2p-jb}; q^{2p})_\infty$

JAMES MC LAUGHLIN AND PETER ZIMMER

Abstract. The present paper examines the phenomenon of coefficients that vanish in a class of infinite products related to those first examined by Hirschhorn, and later by Tang, Baruah and Kaur, and the first author of the present paper. Several infinite families of vanishing coefficient phenomena are found.

In particular, it is shown that if p is a prime in one of several arithmetic progressions modulo 24, then there exist integer values for j, v and w such that if b is any integer (or in same cases, any odd integer, or any even integer), then a result of the following type holds:
If the sequence $\{r_n\}$ is defined by

$$(q^{jb}, q^{2p-jb}; q^{2p})_\infty (q^b, q^{p-b}; q^p)_\infty^3 =: \sum_{n=0}^{\infty} r_n q^n,$$

then $r_{pn+vb^2+wb} = 0$ for all n.

1. Introduction

Results on the vanishing in arithmetic progressions of coefficients in the series expansion of various infinite products go back at least as far as the paper [9] of Richmond and Szekeres, who proved, amongst other results, that if

$$F(q) := \frac{(q^3, q^5; q^8)_\infty}{(q, q^7; q^8)_\infty} =: \sum_{m=0}^{\infty} c_m q^m,$$

then c_{4n+3} is always zero.

Their results were extended by Andrews and Bressoud [2], who proved the following general theorem.

Date: January 14, 2022.

2000 *Mathematics Subject Classification.* Primary:11B65. Secondary: 33D15, 05A19.

Key words and phrases. q-Series, Infinite Products, Infinite q-Products, Vanishing Coefficients, Jacobi triple product identity .

The authors would like thank the anonymous referees for their helpful suggests which have led to improvements in the paper.

Theorem 1.1. *If $1 \leq r < k$ are relatively prime integers of opposite parity and*

$$\frac{(q^r, q^{2k-r}; q^{2k})_\infty}{(q^{k-r}, q^{k+r}; q^{2k})_\infty} =: \sum_{n=0}^{\infty} \phi_n q^n, \tag{1.1}$$

then $\phi_{kn+r(k-r+1)/2}$ is always zero.

Alladi and Gordon [1] proved an even more general theorem.

Theorem 1.2. *Let $1 < m < k$ and let $(s, km) = 1$ with $1 \leq s < mk$. Let $r^* = (k-1)s$ and $r \equiv r^* \pmod{mk}$, with $1 \leq r < mk$.*
Put $r' = \lceil \frac{r^*}{mk} \rceil \pmod{k}$ with $1 \leq r' < k$. Write

$$\frac{(q^r, q^{mk-r}; q^{mk})_\infty}{(q^s, q^{mk-s}; q^{mk})_\infty} = \sum_{n=0}^{\infty} a_n q^n. \tag{1.2}$$

Then $a_n = 0$ for $n \equiv rr' \pmod{k}$.

Alladi and Gordon [1] also proved a companion theorem for odd k, in which the denominator of the infinite product on the right side of (1.2) was replaced with $(-q^s, -q^{mk-s}; q^{mk})_\infty$.

Similar results were obtained in [7] by the first author of the present paper, who proved the following result (and a companion result similar the companion result of Alladi and Gordon [1] mentioned just above).

Theorem 1.3. *Let $k > 1$, $m > 1$ be positive integers. Let $r = sm + t$, for some integers s and t, where $0 \leq s < k$, $1 \leq t < m$ and r and k are relatively prime. Let*

$$\frac{(q^{r-tk}, q^{mk-(r-tk)}; q^{mk})_\infty}{(q^r, q^{mk-r}; q^{mk})_\infty} =: \sum_{n=0}^{\infty} c_n q^n, \tag{1.3}$$

then c_{kn-rs} is always zero.

In [5], Hirschhorn introduced a new class of infinite q-products which have the property that when the product is expanded as a series in q, then the coefficients in one or more arithmetic progressions vanish. More precisely, he proved the following.

Let the sequences $\{a_n\}$ and $\{b_n\}$ be defined by

$$\sum_{n=0}^{\infty} a_n q^n := (-q, -q^4; q^5)_\infty (q, q^9; q^{10})_\infty^3,$$
$$\sum_{n=0}^{\infty} b_n q^n := (-q^2, -q^3; q^5)_\infty (q^3, q^7; q^{10})_\infty^3.$$

Then $a_{5n+2} = a_{5n+4} = b_{5n+1} = b_{5n+4} = 0$.

Following this, a number of similar results were given in [10] by Tang, who proved, for example, that if the sequences $\{a_2(n)\}$, $\{b_2(n)\}$, $\{a_3(n)\}$ and $\{b_3(n)\}$ are defined by

$$\sum_{n=0}^{\infty} a_2(n)q^n := (-q,-q^4;q^5)_\infty^3(q^2,q^8;q^{10})_\infty,$$
$$\sum_{n=0}^{\infty} b_2(n)q^n := (-q^2,-q^3;q^5)_\infty^3(q^4,q^6;q^{10})_\infty,$$
$$\sum_{n=0}^{\infty} a_3(n)q^n := (-q,-q^4;q^5)_\infty^3(q^3,q^7;q^{10})_\infty,$$
$$\sum_{n=0}^{\infty} b_3(n)q^n := (-q^2,-q^3;q^5)_\infty^3(q,q^9;q^{10})_\infty,$$

then $a_2(5n+4) = b_2(5n+1) = a_3(5n+3) = a_3(5n+4) = b_3(5n+3) = b_3(5n+4) = 0$.

Baruah and Kaur proved a number of similar results in [3], such as the following. Let the sequences $\{k_n\}$, $\{l_n\}$, $\{u_n\}$ and $\{v_n\}$ be defined by

$$\sum_{n=0}^{\infty} k_nq^n := (q,q^4;q^5)_\infty(q,q^9;q^{10})_\infty^3, \tag{1.4}$$
$$\sum_{n=0}^{\infty} l_nq^n := (q^2,q^3;q^5)_\infty(q^3,q^7;q^{10})_\infty^3,$$
$$\sum_{n=0}^{\infty} u_nq^n := (q,q^4;q^5)_\infty^3(q^3,q^7;q^{10})_\infty,$$
$$\sum_{n=0}^{\infty} v_nq^n := (q^2,q^3;q^5)_\infty^3(q,q^9;q^{10})_\infty.$$

Then $k_{5n+4} = l_{5n+4} = u_{5n+4} = v_{5n+3} = 0$.

In [8], the first author proved that such results (usually) exist in families. One such example is contained in the following theorem.

Theorem 1.4. *For $b \in \{1,2,\dots,9,10\}$ define the sequence $\{r_n\}$ by*

$$(q^{8b},q^{11-8b};q^{11})_\infty^3(q^{11-b},q^{11+b};q^{22})_\infty =: \sum_{n=-756}^{\infty} r_nq^n. \tag{1.5}$$

(i) For all n, $r_{11n+6b^2+b} = 0$.
(ii) In addition, if $b \in \{1,3,5,7,9\}$, then $r_{11n+4b^2+b} = 0$ for all n.

At the end of the paper [10], Tang posed the problem of finding triples (r,s,t) such that if the sequences $\{a_{r,s,t}(n)\}$, $\{b_{r,s,t}(n)\}$ are defined by

$$\sum_{n=0}^{\infty} a_{r,s,t}(n)q^n := (-q^r,-q^{t-r};q^t)_\infty^3(q^s,q^{2t-s};q^{2t})_\infty, \tag{1.6}$$

$$\sum_{n=0}^{\infty} b_{r,s,t}(n)q^n := (-q^r, -q^{t-r}; q^t)_\infty (q^s, q^{2t-s}; q^{2t})_\infty^3,$$

then these sequences vanish in one or more arithmetic progressions modulo t. Tang stated a number of results for $t = 7$ and $t = 11$.

In the present paper we further consider the question of coefficients vanishing in arithmetic progressions in the series expansion of infinite products of the form

$$\prod(q) := (q^b, q^{p-b}; q^p)_\infty^3 (q^{jb}, q^{2p-jb}; q^{2p})_\infty. \tag{1.7}$$

Here p is taken to be an odd prime, so as to ensure it is relatively prime to the parameter b, thus making solving certain congruences more straightforward. It is shown that there are infinite families of primes p for which the coefficients in the series expansion of $\prod(q)$ vanish in one or more arithmetic progressions modulo p.

An example of a result from the present paper is the following:

Theorem 1.5. *Let p be a prime of the form $p = 24t + 11$, so that $p = 2U^2 + 3V^2$ for positive integers U and V, and $2|U$. Let h and g be any integers such that*

$$hU + 3gV = 1, \tag{1.8}$$

and set $x = 3g + U$ and $y = 3(h - V)$. Let

$$j = 2x(x - 3g) - y\left(h - \frac{y}{3}\right) = 2xU - yV.$$

(i) Let v and w $(0 \le v, w \le p - 1)$ be defined by

$$v \equiv -xV^{-1} \pmod{p}, \tag{1.9}$$

$$w \equiv \frac{j + \chi p + 3}{2} \pmod{p}, \text{ where } \chi = \begin{cases} 0, & j \text{ is odd}, \\ 1, & j \text{ is even}. \end{cases}$$

Let b be any integer and let the sequence $\{r_n\}$ be defined by

$$(q^b, q^{p-b}; q^p)_\infty^3 (q^{jb}, q^{2p-jb}; q^{2p})_\infty = \sum_{n=0}^{\infty} r_n q^n, \tag{1.10}$$

then $r_{pn+vb^2+wb} = 0$ for all integers n.

(ii) Let w be as above and let v $(0 \le v \le p - 1)$ be defined by

$$v \equiv -y(2U)^{-1} \pmod{p}.$$

If y is even, then $r_{pn+vb^2+wb} = 0$ for all integers n and any integer b. If y is odd, then $r_{pn+vb^2+wb} = 0$ for all integers n and any even *integer b.*

Similar theorems are proved for primes p, where $p \pmod{24} \in \{5, 7, 11\}$.

Remark: Equation (1.8) has infinitely many solutions in integers (h, g) since $\gcd(U, 3V) = 1$. Each will lead to a different formulation of the conclusions at (i) and (ii) in the theorem. However, each such statement, where

b is some integer and j, v and w are derived from the particular solution (h, g) chosen for (1.8), may be shown to be equivalent to one of p "fundamental solutions", where $b, j, v, w \in \{0, 1, 2, \dots, p-1\}$.

2. Preliminaries

Observe that multiplying the quantities in (1.7) by infinite products of the forms $(q^p; q^p)_\infty$ or $(q^{2p}; q^{2p})_\infty$ will not have any effect on coefficients that vanish in an arithmetic progression modulo p, and our method of proof involves multiplying the quantities in (1.7) by such infinite products, so as to convert these products into products of Jacobi triple products.

For space saving reasons we will frequently use the notation

$$\langle a; q^j\rangle_\infty$$

to represent the triple product $(a, q^j/a, q^j; q^j)_\infty$ more compactly.

The main tool used to deal the part of the product consisting of a Jacobi triple product cubed is the extended quintuple product formula (see Cao [4, Eq. (3.2)] or Mc Laughlin [6, Eq. (4.6)]):

$$\begin{aligned}
(2.1)\quad & \langle -qa; q^2\rangle_\infty \langle -qb; q^2\rangle_\infty \langle -qc; q^2\rangle_\infty = \left\langle -\frac{q^2 a}{c}; q^4\right\rangle_\infty \\
&\Bigg\{ \left\langle -\frac{q^6 ac}{b^2}; q^{12}\right\rangle_\infty \langle -q^3 abc; q^6\rangle_\infty + qb \left\langle -\frac{q^2 ac}{b^2}; q^{12}\right\rangle_\infty \langle -q^5 abc; q^6\rangle_\infty \\
&\quad + q^4 b^2 \left\langle -\frac{ac}{q^2 b^2}; q^{12}\right\rangle_\infty \langle -q^7 abc; q^6\rangle_\infty \Bigg\} + \frac{q^2 a}{b}\left\langle -\frac{q^4 a}{c}; q^4\right\rangle_\infty \\
&\Bigg\{ \left\langle -\frac{q^{12} ac}{b^2}; q^{12}\right\rangle_\infty \langle -q^3 abc; q^6\rangle_\infty + \frac{b}{q}\left\langle -\frac{q^8 ac}{b^2}; q^{12}\right\rangle_\infty \langle -q^5 abc; q^6\rangle_\infty \\
&\quad + b^2 \left\langle -\frac{q^4 ac}{b^2}; q^{12}\right\rangle_\infty \langle -q^7 abc; q^6\rangle_\infty \Bigg\}.
\end{aligned}$$

To obtain the special case useful for the present purpose, replace q with $q^{p/2}$ and replace a, b and c with $-q^{p/2-b}$, to get (after some elementary q-product manipulations) that

$$\begin{aligned}
(2.2)\quad & \left\langle q^b; q^p\right\rangle_\infty^3 = \\
& \left\langle q^{3b}; q^{3p}\right\rangle_\infty \Big\{ \langle -q^p; q^{2p}\rangle_\infty \langle -q^{3p}; q^{6p}\rangle_\infty + q^p \langle -1; q^{2p}\rangle_\infty \langle -1; q^{6p}\rangle_\infty \Big\} \\
&\quad + \Big\{ -q^b \left\langle q^{p+3b}; q^{3p}\right\rangle_\infty + q^{2b}\left\langle q^{p-3b}; q^{3p}\right\rangle_\infty \Big\} \\
&\quad \times \Big\{ \langle -1; q^{2p}\rangle_\infty \langle -q^{2p}; q^{6p}\rangle_\infty + \langle -q^p; q^{2p}\rangle_\infty \langle -q^p; q^{6p}\rangle_\infty \Big\}.
\end{aligned}$$

For ease of use, we state two equivalent forms of the Jacobi triple product identity as

$$\sum_{n=-\infty}^{\infty}(-z)^n q^{n^2}=(zq,q/z,q^2;q^2)_\infty, \tag{2.3}$$

$$\sum_{n=-\infty}^{\infty}(-z)^n q^{n(n-1)/2}=(z,q/z,q;q)_\infty. \tag{2.4}$$

Both forms are used frequently to expand a Jacobi triple as an infinite bilateral series or to go in the reverse direction; sometimes one form is used, sometimes the other, and it is simpler to have both forms available for easy reference.

3. Idea behind the proofs

The results in this section were derived firstly as a consequence of brute-force searches. These in turn provided sufficient data to enable conjectures to be made, conjectures which were then subsequently proved.

Suppose that it is desired that a prime $p>3$ and integers j, w and v may be found so that if the sequence $\{r_n\}$ is defined by

$$(q^b,q^{p-b};q^p)_\infty^3(q^{jb},q^{2p-jb};q^{2p})_\infty=\sum_{n=0}^{\infty}r_nq^n, \tag{3.1}$$

then $r_{pn+vb^2+wb}=0$, for all n and all integers b, or possibly all even/odd integers b. Observe that this is trivially true if $p|b$, since $(q^b,q^{p-b};q^p)_\infty=0$ in this case, so in what follows it is assumed that $\gcd(b,p)=1$. The situation where j is a multiple of p is also ignored, as then the problem of coefficients vanishing in a progression modulo p devolves to that for the simpler product $(q^b,q^{p-b};q^p)_\infty^3$.

By (2.2), $r_{pn+vb^2+wb}=0$ for all integers n and some integer b, if the coefficients in the series expansions of $\left\langle q^{jb};q^{2p}\right\rangle_\infty\left\langle q^{3b};q^{3p}\right\rangle_\infty$ and

$$\left\langle q^{jb};q^{2p}\right\rangle_\infty\left\{-q^b\left\langle q^{p+3b};q^{3p}\right\rangle_\infty+q^{2b}\left\langle q^{p-3b};q^{3p}\right\rangle_\infty\right\}$$

also vanish in the arithmetic progression $vb^2+wb \pmod p$. Note that it may happen that the coefficients in the series expansions of

$$-q^b\left\langle q^{p+3b};q^{3p}\right\rangle_\infty\left\langle q^{jb};q^{2p}\right\rangle_\infty \qquad\text{and}\qquad q^{2b}\left\langle q^{p-3b};q^{3p}\right\rangle_\infty\left\langle q^{jb};q^{2p}\right\rangle_\infty$$

vanish separately in the arithmetic progression $vb^2+wb \pmod p$.

By the Jacobi triple product identity (2.4),

$$\left\langle q^{jb};q^{2p}\right\rangle_\infty\left\langle q^{3b};q^{3p}\right\rangle_\infty=\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}q^{pn(n-1)+jbn+3bm+3pm(m-1)/2}. \tag{3.2}$$

From the exponent of q it may be seen that to get powers of q in the progression $vb^2 + wb \pmod p$, it is necessary and sufficient that m and n be restricted so that

$$\begin{aligned} jbn + 3bm &\equiv vb^2 + wb \pmod p, \\ \implies jn + 3m &\equiv vb + w \pmod p. \end{aligned} \tag{3.3}$$

The second congruence implies a pair of equations involving m and n:

$$\begin{aligned} jn + 3m &= vb + w + pr, \\ (j-p)n + 3m &= vb + w + ps, \end{aligned} \tag{3.4}$$

for some integers r and s. It is necessary to solve the last pair of equations for m and n, and while it is clear that $n = r - s$, solving for m introduces a factor of 1/3:

$$\begin{aligned} m &= \frac{1}{3}(pr + j(-r+s) + bv + w), \\ n &= r - s. \end{aligned} \tag{3.5}$$

If r is unrestricted, then it is necessary to restrict s to a particular arithmetic progression modulo 3 (to get integral m), so we write $s = 3k + u$, where $u \in \{-1, 0, 1\}$ to get

$$\begin{aligned} m &= \frac{1}{3}(pr + j(3k - r + u) + bv + w), \\ n &= -3k + r - u \end{aligned} \tag{3.6}$$

Subsequent calculations are best performed using a computer algebra system such as *Mathematica*.

When these substitutions are made in the right side of (3.2), one gets an unrestricted bilateral double sum over the integer variables r and k. However, there is now an additional complication in that the exponent now contains a "cross-term" involving the product rk, namely, $p(-6-j^2+jp)rk$, which inhibits the separation of the double sum into a product of single sums, one over r and the other over k (one of which, for our purposes, we would like to sum to zero).

The next task is to look for an invertible transformation of the form

$$\begin{aligned} k &\to gk + hr, \\ r &\to xk + yr, \end{aligned} \tag{3.7}$$

which will remove the cross terms. Note that the condition $gy - hx = \pm 1$ ensures invertibility. Further, since k and r range over all the integers, so that k (respectively r) may be replaced with $-k$ (respectively $-r$), it is sufficient to restrict to transformations with either $gy - hx = 1$ or $gy - hx = -1$, so here we make the latter choice.

Suppose that after the transformations at (3.6) and (3.7) are made, the exponent of q on the right side of (3.2) becomes

$$\alpha_0 + \alpha_k k + \alpha_r r + \alpha_{kr} kr + \alpha_{k^2} k^2 + \alpha_{r^2} r^2, \tag{3.8}$$

where each of the α's are polynomials in p, j, g, h, u, v, w, x and y (although it may be seen from (3.6) that the expressions for α_{k^2} and α_{r^2} do not involve u, v and w).

Then one can check (once again, most easily by performing the operations using a computer algebra system such as *Mathematica*) that

$$\begin{aligned}(3.9)\quad \alpha_{kr} &= \alpha_{kr}(p,j,g,h,x,y)\\ &= \frac{p}{3}\bigg\{(6+j^2)(3g-x)(3h-y)+p(3j(hx+gy)+(p-2j)xy)\bigg\},\end{aligned}$$

so that if $\alpha_{kr}(p,j,g,h,x,y)=0$ for particular values of g, h, x and y, then $\alpha_{kr}(p,j,-h,g,-y,x)=0$ also. These two solutions may be termed "inequivalent", as it will be seen that they may lead to vanishing coefficients in different arithmetic progressions (solutions arising from replacing all of g, h, x and y (or various pairs of them) by their negatives may be ignored as these correspond to replacing k and/or r by their negatives).

One also finds that

$$\begin{aligned}(3.10)\qquad \alpha_{k^2} &= \frac{p}{6}((6+j^2)(3g-x)^2+2jp(3g-x)x+p^2x^2),\\ \alpha_{r^2} &= \frac{p}{6}((6+j^2)(3h-y)^2+2jp(3h-y)y+p^2y^2).\end{aligned}$$

At this point it is not obvious how to solve $\alpha_{kr}(p,j,g,h,x,y)=0$ generally, or even to find an infinite family of solutions, so a search for specific solutions was performed experimentally instead, to try to see some patterns.This involved searching over primes $p \leq 2400$ and integers j, $1 \leq j \leq p-1$ and checking if there was a quadruple of integers g, h, x and y with $hx-gy=1$ such that $\alpha_{kr}(p,j,g,h,x,y)=0$

In all of the cases found for which $\alpha_{kr}(p,j,g,h,x,y)=0$, it was found that α_{k^2} and α_{r^2} were rational multiples of p^2. More precisely, it appeared that $(\alpha_{k^2},\alpha_{r^2})/p^2$ is one of the pairs

$$(3.11)\qquad (1/2,3),(1,3/2),(3/2,1),(3,1/2).$$

In fact, it can be shown that these are the only possibilities for $(\alpha_{k^2},\alpha_{r^2})/p^2$ under the circumstances described above. More precisely, the following is true.

Proposition 3.1. *Suppose that the transformations* (3.6) *and* (3.7) *are applied to the right side of* (3.2)*, so that the exponent of* q *becomes as described at* (3.8) *above. Suppose, in addition to* $\alpha_{kr}=0$*, that* $\alpha_{k^2}=zp^2$*, for some rational* z*. Then* $\alpha_{r^2}=3p^2/(2z)$*, and*

$$(3.12)\qquad z\in\left\{\frac{1}{2},1,\frac{3}{2},3\right\}.$$

Proof. With α_{kr} and α_{k^2} as at (3.9) and (3.10), one can check (possibly with the aid of a computer algebra system and using the fact that $hx-gy=1$)

that solving the pair of equations $\alpha_{kr} = 0$, that $\alpha_{k^2} = zp^2$ for j and p leads to

$$p = \frac{1}{z}(x-3g)^2 + \frac{2z}{3}(3h-y)^2, \tag{3.13}$$
$$j = \frac{x}{z}(x-3g) - \frac{2zy}{3}(3h-y).$$

(Note that the "trivial" solutions involving $p = 0$ are ignored, since p is assumed here to be a prime.) If these values are substituted into the expression for α_{r^2} at (3.10), it is found that $\alpha_{r^2} = 3p^2/(2z)$, as claimed.

Since the coefficients of m, m^2, n and n^2 in the exponent of q at (3.2) are integers or half integers, the same must be true of the coefficients of k^2 ($\alpha_{k^2} = zp^2$) and r^2 ($\alpha_{r^2} = 3p^2/(2z)$), and these facts limit the possibilities for z.

Firstly, the situation where either of $x - 3g$ or $y - 3h$ is equal to zero is ignored, since it is easily seen to imply that j is then an integer multiple of p, a case not being considered by the remark following Equation (3.1).

We set the rational number $z = N/D$ for relatively prime integers N and D. It is a fact that $p \nmid N$, since the integrality or half-integrality of $\alpha_{k^2} = zp^2$ and $\alpha_{r^2} = 3p^2/(2z)$ then imply that $D \in \{1, 2\}$ and $N \in \{p, 3p, p^2, 3p^2\}$ and substituting $z = N/D$ with each of these possibilities into the expression for p at (3.13) at leads to a contradiction. A similar argument shows $p \nmid D$.

Finally, once again considering the integrality or half-integrality of $\alpha_{k^2} = zp^2$ and $\alpha_{r^2} = 3p^2/(2z)$, this time using the facts that $p \nmid N$ and $p \nmid D$, one gets that $D \in \{1, 2\}$ and $N \in \{1, 3\}$, giving the claimed values for z. □

For later use, we record the following identities.

Proposition 3.2. *Let $z = N/D$ be as in the previous proposition. Then*

$$6 + j^2 = p\left(\frac{Dx^2}{N} + \frac{2Ny^2}{3D}\right), \tag{3.14}$$
$$p - j = \frac{3Dg(3g-x)}{N} + \frac{2hN(3h-y)}{D},$$
$$(3g-x)j + px = \frac{2N(3h-y)}{D},$$
$$(3h-y)j + py = -\frac{3D(3g-x)}{N}.$$

Proof. These all follow from substituting for p and j from (3.13), possibly using the fact that $hx - gy = 1$. □

Suppose that the cross-terms involving rk have been eliminated ($\alpha_{kr} = 0$). Then the next task is to check if integers w and v may be found so that the coefficients in the series expansions of $\langle q^{jb}; q^{2p}\rangle_\infty \langle q^{3b}; q^{3p}\rangle_\infty$ vanish in the arithmetic progression $vb^2 + wb \pmod p$. With our previous notation, suppose that after the transformations at (3.6) and (3.7) are made and the pair of equations $\alpha_{kr} = 0$, $\alpha_{k^2} = zp^2$ are solved for j and p, the sum that contains only powers of q in the arithmetic progression $vb^2 + wb \pmod{p}$ is represented formally by

$$F_1(q) := (-1)^{\psi_1} q^{\alpha_0} \sum_{k\in\mathbb{Z}} (-1)^{\beta_1 k} q^{\alpha_k k + \alpha_{k^2} k^2} \sum_{r\in\mathbb{Z}} (-1)^{\rho_1 r} q^{\alpha_r r + \alpha_{r^2} r^2}. \tag{3.15}$$

It may be seen from (2.3) that if β_1 is odd and α_k is an odd integer multiple of α_{k^2} (or ρ_1 is odd and α_r is an odd integer multiple of α_{r^2}), then $F_1(q) = 0$, so that all coefficients of $\left\langle q^{jb}; q^{2p}\right\rangle_\infty \left\langle q^{3b}; q^{3p}\right\rangle_\infty$ vanish in the arithmetic progression $vb^2 + wb \pmod{p}$. The idea is to try to make choices for v and w so that one of these two pairs of conditions are satisfied.

For the above reasons, we examine the values taken by α_k and α_r at this stage, and also consider how $m+n$ has transformed (since the $(-1)^{\beta_1 k}$ and $(-1)^{\rho_1 r}$ terms derive from the original $(-1)^{m+n}$ term).

Proposition 3.3. *With the notation of* (3.15) *and with the additional substitution $w = (j+3+\chi p)/2$, where $\chi = (j+1) \pmod{2}$, one has*

$$\begin{aligned} \frac{\alpha_k}{p} &= b\left(\frac{2}{3}vz(3h-y)+x\right) + \frac{1}{3}p\chi z(3h-y) - \frac{2}{3}puyz - \frac{1}{3}pyz, \\ \frac{\alpha_r}{p} &= b\left(\frac{v(x-3g)}{z}+y\right) - \frac{p\chi(3g-x)}{2z} + \frac{pux}{z} + \frac{px}{2z}, \end{aligned} \tag{3.16}$$

$$\begin{aligned} \beta_1 &= x - 3g + \frac{2}{3}z(3h-y), \\ \rho_1 &= \frac{x-3g}{z} + y - 3h. \end{aligned} \tag{3.17}$$

Proof. At this stage,

$$\begin{aligned} \frac{\alpha_k}{p} = b\left(gjv - \frac{jvx}{3} + \frac{pvx}{3} + x\right) + u\left(\frac{1}{3}(j^2+6)(3g-x) + \frac{jpx}{3}\right) \\ + \frac{1}{6}(3g-x)(j(2w-3)+6) + \frac{1}{6}p(2w-3)x. \end{aligned} \tag{3.18}$$

Since it is desired that α_k be an odd-integer multiple of $\alpha_{k^2} = zp^2$, then the right side of (3.18) should be a rational multiple of p. The term involving u actually is a rational multiple of p in disguised form, as may be seen from the first equation at (3.14). It can also be seen that the $j(2w-3)+6$ factor in the third term becomes j^2+6 (a rational multiple of p, again by the first equation at (3.14)), if $2w-3 = j$. However, this is not solvable for integral w if j is even, so we set $w = (j+3+\chi p)/2$, where $\chi = (j+1) \pmod{2}$.

After making this substitution and also substituting for p and j from (3.13) at particular places on the right side, and/or employing some of the transformations at (3.14) (in particular, using the first of these to replace any occurrence of j^2, but replacing the N/D on the right sides by z), this equation becomes the expression on the right side of the first equation at (3.16). This expression will be a rational multiple of p if v can be chosen so that $2v(3h-y)z/3 \equiv -x \pmod{p}$. By a similar analysis, it is found that α_r is as stated at the second equation at (3.16).

Finally, after all the transformations described, $m+n$ has been transformed as shown:

$$(3.19)\quad \begin{aligned} m+n \to \frac{1}{3}(bv+ju-3u+w) &+ k\left(gj-3g-\frac{jx}{3}+\frac{px}{3}+x\right) \\ &+ r\left(hj-3h-\frac{jy}{3}+\frac{py}{3}+y\right) \\ = \frac{1}{6}(2bv+2ju+j+p\chi-6u+3) &+ k\left(x-3g+\frac{2}{3}z(3h-y)\right) \\ &+ r\left(\frac{x-3g}{z}+y-3h\right), \end{aligned}$$

so that β_1 and ρ_1 are as described at (3.17).

Note that the third and fourth equations at (3.14) were employed to go from the first expression for $m+n$ at (3.19) to the second. □

The next task is to consider the series expansions

(3.20)

$$-q^b\left\langle q^{p+3b};q^{3p}\right\rangle_\infty \left\langle q^{jb};q^{2p}\right\rangle_\infty = -q^b\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}q^{pn(n-1)+jbn+(p+3b)m+3pm(m-1)/2},$$

$$q^{2b}\left\langle q^{p-3b};q^{3p}\right\rangle_\infty \left\langle q^{jb};q^{2p}\right\rangle_\infty = q^{2b}\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}q^{pn(n-1)+jbn+(p-3b)m+3pm(m-1)/2}.$$

As above, replacements are made for m and n so as to obtain just the powers of q in the arithmetic progression $vb^2+wb \pmod{p}$. By reasoning similarly (and taking account of the q^b and q^{2b} terms multiplying the series), this can be seen to entail solving the pairs of equations

$$(3.21)\quad \begin{aligned} jn+3m &= wb+v-1+pr, \\ (j-p)n+3m &= wb+v-1+ps, \end{aligned}$$

and

$$(3.22)\quad \begin{aligned} jn-3m &= wb+v-2+pr, \\ (j-p)n-3m &= wb+v-2+ps, \end{aligned}$$

for m and n. By continuing to follow similar reasoning to that used above, the first pair of equations leads to

$$(3.23)\quad m = \frac{1}{3}(-1+pr+j(3k-r+u')+bv+w),$$

$$n = -3k + r - u',$$

and the second pair to

$$\begin{aligned} m &= \frac{1}{3}(2 - pr + j(-3k + r - u'') - bv - w), \\ n &= -3k + r - u''. \end{aligned} \tag{3.24}$$

As above, replacements of the form $s \to 3k + u'$ (respectively, $s \to 3k + u''$) have been made.

These substitutions are then made on the right sides respectively of (3.20). Since the coefficients of n^2 and m^2 in the exponent of q in these series is the same as those in the exponent of q in the series at (3.2), and since it is the n^2 and m^2 terms that produce the cross-terms rk, then the same transformation (3.7) will remove the rk terms in the two series that are derived from (3.20).

Hence we proceed as follows: substitutions for m and n are made in the first series on the right at (3.20) using the values at (3.23), and likewise in the second series at (3.20) using the values at (3.24). Next, the same linear transformation (3.7) that was applied (in theory) to remove the kr terms in the series deriving from the expansion of $\langle q^{jb}; q^{2p} \rangle_\infty \langle q^{3b}; q^{3p} \rangle_\infty$ is applied to each of the two resulting series (to remove the terms in kr from these series also, and permit each double series to be factored into a series over k and a series over r). The same substitutions for p and j as at (3.13) are applied to actually make the kr terms disappear (recall that these were derived by solving the pair of equations $\alpha_{kr} = 0$, $\alpha_{k^2} = zp^2$). In each case the same values that were previously assigned to w and v (which caused all terms in the arithmetic progression $vb^2 + wb \pmod{p}$ in the series expansion of $\langle q^{jb}; q^{2p} \rangle_\infty \langle q^{3b}; q^{3p} \rangle_\infty$ to vanish) are also assigned in each of these two series.

For reference purposes, the exponent of q in each of the two series at (3.20), after all of these changes have been made, are denoted by

$$\alpha'_0 + \alpha'_k k + \alpha'_r r + \alpha_{k^2} k^2 + \alpha_{r^2} r^2 \tag{3.25}$$

and

$$\alpha''_0 + \alpha''_k k + \alpha''_r r + \alpha_{k^2} k^2 + \alpha_{r^2} r^2, \tag{3.26}$$

respectively. Let

$$\begin{aligned} F_2(q) &:= (-1)^{\psi_2(b)} q^{\alpha'_0} \sum_{k \in \mathbb{Z}} (-1)^{\beta_2 k} q^{\alpha'_k k + \alpha_{k^2} k^2} \sum_{r \in \mathbb{Z}} (-1)^{\rho_2 r} q^{\alpha'_r r + \alpha_{r^2} r^2}, \\ F_3(q) &:= (-1)^{\psi_3(b)} q^{\alpha''_0} \sum_{k \in \mathbb{Z}} (-1)^{\beta_3 k} q^{\alpha''_k k + \alpha_{k^2} k^2} \sum_{r \in \mathbb{Z}} (-1)^{\rho_3 r} q^{\alpha''_r r + \alpha_{r^2} r^2}, \end{aligned} \tag{3.27}$$

be the series derived respectively from the right sides of (3.20) after applying all of the stated transformations and substitutions. Note that all of the powers of q in the series $F_2(q)$ and $F_3(q)$ are, by construction, congruent to $vb^2 + wb \pmod{p}$.

Next one can check for either of two possibilities. The first possibility is that in each case either the series over k or the series over r vanishes identically (for similar reasons to those described after (3.15)).

The second possibility that leads to all coefficients vanishing in the arithmetic progression $vb^2 + wb \pmod{p}$ in the series expansion of the infinite product at (3.1) is if $F_2(q) \neq 0$ but $F_2(q) + F_3(q) = 0$ (this second possibility does not occur in the cases examined in the theorems in the next section).

Other relevant results are summarized in the next proposition.

Proposition 3.4. *With the notation of equations* (3.15)*,* (3.16)*,* (3.17) *and* (3.27)*, we have*

$$\begin{aligned}
\alpha_k &= \alpha'_k = \alpha''_k, \\
\alpha_r &= \alpha'_r = \alpha''_r, \\
\beta_2 &= \beta_1, \\
\rho_2 &= \rho_1, \\
\beta_3 &= x - 3g - \frac{2}{3} z(3h - y), \\
\rho_3 &= -\frac{x - 3g}{z} + y - 3h.
\end{aligned} \tag{3.28}$$

Proof. These all follow as a consequence of the same kind of analysis as was employed in the proof of Proposition 3.4, perhaps assisted by a computer algebra system such as *Mathematica*. □

Remarks: (i) What these results mean is that to apply Propositions 3.1 3.4 in any particular situation so as to get $F_1(q) = F_2(q) = F_3(q) = 0$, all that is necessary is to show that β_1 and β_3 are odd, and that α_k is an odd integer multiple of α_{k^2}.

(ii) In some of the theorems below, it will be seen that the result holds for just odd b or just even b. The reasons will be explained when the phenomenon is encountered.

4. Main Results

The results in this section follow from investigating the possibilities implied by the observation at (3.11).

As mentioned earlier, some of the assertions in this and later proofs are most easily checked with a computer algebra system like *Mathematica*. Most rely on the fact that $hx - gy = 1$ for simplification. We first prove Theorem 1.5 from the introduction.

Proof of Theorem 1.5. Set $z = 1/2$ in Proposition 3.1, so that by (3.13),

$$\begin{aligned}
p &= 2(x - 3g)^2 + 3\left(h - \frac{y}{3}\right)^2, \\
j &= 2x(x - 3g) - y\left(h - \frac{y}{3}\right),
\end{aligned} \tag{4.1}$$

in agreement with the statements of the theorem, if g and x are both odd. Further, from the proof of Proposition 3.1,

$$(\alpha_{kr}, \alpha_{k^2}, \alpha_{r^2}) = \left(0, \frac{1}{2}p^2, 3p^2\right).$$

From (3.17) and (3.28) (and the comment preceding the latter) it follows that

$$\beta_1 = \beta_2 = x-3g+(h-y/3) = U+V, \ \beta_3 = x-3g-(h-y/3) = U-V, \tag{4.2}$$

and thus are all odd (recall U is even and V is odd), as required.

One can also check that

$$\begin{aligned} \frac{\alpha_k}{p} &= b\left(v\left(h-\frac{y}{3}\right)+x\right)+\frac{1}{2}p\chi\left(h-\frac{y}{3}\right)-pu\frac{y}{3}-\frac{1}{2}p\frac{y}{3} \\ &= b\,(vV+x)+\frac{1}{2}p\chi V-pu\frac{y}{3}-\frac{1}{2}p\frac{y}{3}. \end{aligned} \tag{4.3}$$

With the choice for v stated in the theorem, $b(vV+x)$ is a multiple of p, as is $puy/3$, since $3|y$. Since $j = 2xU - yV$ and V is odd, then j and y have the same parity. If y is odd then j is odd and $\chi = 0$, and $py/6$ is an odd integer multiple of $p/2$. If y is even, then j is even, so that $\chi = 1$, $py/6$ is an integer multiple of p and $p\chi V/2 = pV/2$ is an odd integer multiple of $p/2$. Thus in either case, α_k is an odd integer multiple of $p^2/2$. Thus α_k is an odd integer multiple of α_{k^2} and so $F_1(q) = 0$.

Upon turning to $F_2(q)$, since $\beta_2 = \beta_1$ (so also odd), and $\alpha'_k = \alpha_k$ (and hence also an odd integer multiple of α_{k^2}), $F_2(q) = 0$.

Since $\beta_3 = U - V$ (again odd), and $\alpha''_k = \alpha_k$ (and hence also an odd integer multiple of α_{k^2}), then $F_3(q) = 0$. This completes the proof of (i).

For (ii), we start by setting $z = 3$ in Proposition 3.1 and its proof. This gives

$$\begin{aligned} p &= \frac{1}{3}(x'-3g')^2 + 2(3h'-y')^2, \\ j &= \frac{x'}{3}(x'-3g') - 2y'(3h'-y'), \\ \beta_1 &= x'-3g'+2(3h'-y'), \\ \beta_3 &= x'-3g'-2(3h'-y'), \\ \frac{\alpha_k}{p} &= b(2v(3h'-y')+x') + p\chi(3h'-y') - 2puy' - py'. \end{aligned} \tag{4.4}$$

The $'$ marks are used to distinguish the values that appear here from those in part (i). To get the values for p and j that appear in part (i), the following substitutions are made:

$$g' \to h, \qquad h' \to -g, \qquad x' \to y, \qquad y' \to -x.$$

With these changes, p and j take the forms stated at (4.1),

$$\begin{aligned}\beta_1 &= 2(x-3g) - 3\left(h - \frac{y}{3}\right) = 2U - 3V,\\ \beta_3 &= -2(x-3g) - 3\left(h - \frac{y}{3}\right) = -2U - 3V,\\ \frac{\alpha_k}{p} &= b(2v(x-3g)+y) + p\chi(x-3g) + 2pux + px,\\ &= b(2vU+y) + p\chi U + 2pux + px.\end{aligned} \tag{4.5}$$

Once again, $\beta_1 = \beta_2$ and β_3 are odd (recall that U is even and V is odd), and it remains to show that α_k is an odd-integer multiple of $\alpha_{k^2} = 3p^2$, and that the stated dependence on the parity of y holds. The proof this time is slightly more technical than the proof of (i). It may be seen from the second equation at (3.14) that $3|(p-j)$, since $3|y$ $(y = 3(h-V))$. Since p has the form $24t+11$, then $j \equiv 2 \pmod 3$. Hence from the first equation at (3.6), $3|(2u+bv+w)$, or $u = -2w+bv+3T$, for some integer T. Upon substituting for w from (1.9),

$$\begin{aligned} &u = -j - 3 - \chi p + bv + 3T, \\ \implies &\frac{\alpha_k}{p} = b(2v(px+U)+y) - (2j+5)px + 6pTx + p\chi(U-2px).\end{aligned} \tag{4.6}$$

With the value stated for v in the theorem, $p|(2vU+y)$. From the stated values for U and p,

$$px + U \equiv (2x^3 + x) = x(2x^2+1) \equiv 0 \pmod 3.$$

Thus $6p|(2v(px+U)+y)$ if y is even, while $2v(px+U)+y$ is odd if y is odd and hence is divisible by just $3p$ in that case. Since x is odd and $j \equiv 2 \pmod 3$, $(2j+5)px$ is an odd multiple of $3p$. Since U is even, $U-2px$ is even, and since $3|U+px$ from above, then $3|U-2px$, and $p\chi(U-2px)$ is a multiple of $6p$, as is of course $6pTx$. Hence α_k is an odd multiple of $3p^2 = \alpha_{k^2}$ for all b if y is even, and for all even b if y is odd, and the proof of (ii) is complete. □

As an example we consider $p = 59$. We ignore 11, since vanishing coefficient results for this prime have been given elsewhere (see the introduction).

Example 1. *Let* $p = 59 = 24(2)+11 = 2(4^2)+3(3^2)$. *So* $(U,V) = (4,3)$. *Let* $(h,g) = (-2,1)$ *be a solution to*

$$hU + 3gV = h(4) + 3g(3) = 1.$$

Then $x = U + 3g = 4 + 3(1) = 7$, *and* $y = 3(h-V) = 3(-2-3) = -15$. *Next,* $j = 2xU - yV = 2(7)(4) - (-15)(3) = 101$. *Since j is odd,* $w = (j+3)/2 = 52$. *Since* $V^{-1} \pmod{59} = 20$, *then* $-xV^{-1} = -(7)(20) \equiv 37 \pmod{59}$, *so* $v = 37$. *Hence, by the theorem, if the sequence* $\{r_n\}$ *is defined by*

$$(q^b, q^{59-b}; q^{59})_\infty^3 (q^{101b}, q^{118-101b}; q^{118})_\infty = \sum_{n=0}^{\infty} r_n q^n,$$

then $r_{59n+37b^2+52b} = 0$*, for all integers* n *and* b*.*

Upon turning to part (ii), $y = -15$ *(odd) and* $U = 4$*,*

$$-y(2U)^{-1} = 15(8)^{-1} \equiv 24 \pmod{59},$$

and $r_{59n+24b^2+52b} = 0$*, for all integers* n *and all even integers* b*.*

Remark: In the theorem, p was restricted to be a prime since all primes p in the arithmetic progressions 11 (mod 24) have the representation $p = 2U^2 + 3V^2$ (a key requirement of the proof), and also to allow a factor of b to be dropped from the congruence (3.3), but in fact this restriction may be relaxed to non-prime p with the representation $p = 2U^2 + 3V^2$, with U even, and b is restricted so that $\gcd(b, p) = 1$.

It should be noted that there may not be any solutions to

$$2U^2 + 3V^2 = 24t + 11$$

with U even when $24t + 11$ is composite (there is no solution for $t = 12$ and $24(12) + 11 = 299$, for example).

Example 2. *Let* $p = 35 = 2(2^2) + 3(3^2)$*. So* $(U, V) = (2, 3)$*. Let* $(h, g) = (-4, 1)$ *be a solution to*

$$hU + 3gV = h(2) + 3g(3) = 1.$$

Then $x = U+3g = 2+3(1) = 5$*, and* $y = 3(h-V) = 3(-4-3) = -21$*. Next,* $j = 2xU - yV = 2(5)(2) - (-21)(3) = 83$*. Since* j *is odd,* $w = (j + 3)/2 = 43 \equiv 8 \pmod{35}$*. Since* $V^{-1} \pmod{35} = 12$*, then* $-xV^{-1} = -(5)(12) \equiv 10 \pmod{35}$*, so* $v = 10$*. Thus if* $\gcd(b, 35) = 1$ *and the sequence* $\{r_n\}$ *is defined by*

$$(q^b, q^{35-b}; q^{35})^3_\infty (q^{83b}, q^{70-83b}; q^{70})_\infty = \sum_{n=0}^{\infty} r_n q^n,$$

then $r_{35n+10b^2+8b} = 0$*, for all integers* n *and any integer* b *relatively prime to 35.*

For the arithmetic progression predicted by part (ii) of the theorem, $y = -21$ *(odd),* $U = 2$*,*

$$-y(2U)^{-1} = 21(4)^{-1} \equiv 14 \pmod{35},$$

and $r_{35n+14b^2+8b} = 0$*, for all integers* n *and any even integer* b *that is relatively prime to 35.*

Remark: There is another solution to $2U^2 + 3V^2 = 35$, namely $U = 4$ and $V = 1$. By following the same steps as in the previous example (with $h = -2$ and $g = 3$), one gets the following result.

Example 3. *If* $\gcd(b, 35) = 1$ *and the sequence* $\{r_n\}$ *is defined by*

$$(q^b, q^{35-b}; q^{35})^3_\infty (q^{113b}, q^{70-113b}; q^{70})_\infty = \sum_{n=0}^{\infty} r_n q^n,$$

then $r_{35n+22b^2+23b} = 0$*, for all integers* n *and any integer* b *relatively prime to 35, and* $r_{35n+23b^2+23b} = 0$*, for all integers* n *and any even integer* b *relatively prime to 35.*

By similar reasoning the following theorem holds.

Theorem 4.1. *Let* p *be a prime of the form* $p = 24t + 5$*, so that* $p = 2U^2 + 3V^2$ *for positive integers* U *and* V*, both odd. Let* h *and* g *be integers such that*

$$hU + 3gV = 1, \tag{4.7}$$

with h *even and* g *odd, and set* $x = 3g + U$ *and* $y = 3(h - V)$*. Let*

$$j = 2x(x - 3g) - y\left(h - \frac{y}{3}\right) = 2xU - yV.$$

Let v *and* w $(0 \leq v, w \leq p - 1)$ *be defined by*

$$\begin{aligned} v &\equiv -y(2U)^{-1} \pmod{p}, \\ w &\equiv \frac{j+3}{2} \pmod{p}. \end{aligned} \tag{4.8}$$

Let b *be any odd integer and let the sequence* $\{r_n\}$ *be defined by*

$$(q^b, q^{p-b}; q^p)_\infty^3 (q^{jb}, q^{2p-jb}; q^{2p})_\infty = \sum_{n=0}^{\infty} r_n q^n. \tag{4.9}$$

Then $r_{pn+vb^2+wb} = 0$ *for all integers* n*.*

Remarks: (i) Note that x is even and y is odd, and thus that j is odd. This is the reason for the simplified definition of w in Theorem 4.1, compared with that in Theorem 1.5.

(ii) Note that setting $z = 1/2$ in Proposition 3.1 does produce any results in this case, since from (4.2), each of β_1, β_2 and β_3 is even.

Proof. The proof very closely follows that of part (ii) of Theorem 1.5, in that the $z = 3$ case of Proposition 3.1 is used. The differences in this case are that U is odd and that the parity of g, h, x and y is fixed. Note from (4.5) that β_1, β_2 and β_3 are odd, as needed.

To get that α_k is an odd integer multiple of $\alpha_{k^2} = 3p^2$, most of the argument is the same as before. In (4.6), $\chi = 0$ since j is odd. The term $(2j + 5)px$ is a multiple of $6p$ since, as noted earlier, $j \equiv p \pmod{3}$ and in this case $p \equiv 2 \pmod{3}$. Thus it is necessary that $b(2v(px + U) + y)$ be an odd multiple of $3p$, and from what has been shown earlier, it is necessary and sufficient that b be odd. □

Example 4. *Let* $p = 53 = 24(2) + 5 = 2(5^2) + 3(1^2)$*. So* $(U, V) = (5, 1)$*. Let* $(h, g) = (2, -3)$ *be a solution to*

$$hU + 3gV = h(5) + 3g(1) = 1.$$

Then $x = U + 3g = 5 + 3(-3) = -4$, *and* $y = 3(h - V) = 3(2 - 1) = 3$. *Next,* $j = 2xU - yV = 2(-4)(5) - (3)(1) = -43$. *Since* j *is odd,* $w = (j + 3)/2 = -20$. *Since* $y = 3$ *and* $U = 5$,

$$-y(2U)^{-1} = -3(10)^{-1} \equiv 5 \pmod{53},$$

so $v = 5$. *Thus if the sequence* $\{r_n\}$ *is defined by*

$$(q^b, q^{53-b}; q^{53})^3_\infty (q^{-43b}, q^{106+43b}; q^{106})_\infty = \sum_{n=0}^{\infty} r_n q^n, \tag{4.10}$$

then $r_{53n+5b^2-20b} = 0$, *for all integers* n *and all odd integers* b.

Remark: In the theorem it was specified that $v, w \in \{0, 1, \ldots, p-1\}$, but since they are just relevant to the arithmetic progression $pn + vb^2 + wb$, it is clear that any value for v or w in the same residue class $\pmod{p}$, will work, so we chose -20 (the direct output from $(j + 3)/2 = -20$) for w in the example instead of 33.

We next turn to primes $p \equiv 1 \pmod{24}$.

Theorem 4.2. *Let* p *be a prime of the form* $p = 24t+1$, *so that* $p = U^2+6V^2$ *for integers* U *(odd) and* V *(even). Let* h *and* g *be integers such that*

$$hU + 3gV = 1, \tag{4.11}$$

and set $x = 3g + U$ *and* $y = 3(h - V)$. *Let*

$$j = x(x - 3g) - 2y\left(h - \frac{y}{3}\right) = xU - 2yV.$$

(i) Let v *and* w $(0 \leq v, w \leq p - 1)$ *be defined by*

$$v \equiv -x(2V)^{-1} \pmod{p}, \tag{4.12}$$

$$w \equiv \frac{j + \chi p + 3}{2} \pmod{p}, \text{ where } \chi = \begin{cases} 0, & j \text{ is odd,} \\ 1, & j \text{ is even.} \end{cases}$$

Let b *be an integer and let the sequence* $\{r_n\}$ *be defined by*

$$(q^b, q^{p-b}; q^p)^3_\infty (q^{jb}, q^{2p-jb}; q^{2p})_\infty = \sum_{n=0}^{\infty} r_n q^n. \tag{4.13}$$

If x *is even then* $r_{pn+vb^2+wb} = 0$ *for all integers* n *and all integers* b, *while if* x *is odd, then* $r_{pn+vb^2+wb} = 0$ *for all integers* n *and all even integers* b.

(ii) Let w *be as above and let* v $(0 \leq v \leq p - 1)$ *be defined by*

$$v \equiv -y(U)^{-1} \pmod{p}).$$

Then $r_{pn+vb^2+wb} = 0$ *for all integers* n *and any integer* b.

Proof. Set $z = 1$ in Proposition 3.1, so that by (3.13),

$$p = (x - 3g)^2 + 6\left(h - \frac{y}{3}\right)^2, \tag{4.14}$$

$$j = x(x - 3g) - 2y\left(h - \frac{y}{3}\right),$$

giving the values for p and j in the statements of the theorem. Further, from the proof of the proposition,

$$(\alpha_{kr}, \alpha_{k^2}, \alpha_{r^2}) = \left(0, p^2, \frac{3}{2}p^2\right).$$

From (3.17) and (3.28) (and the comment preceding the latter) it follows that

(4.15)

$$\beta_1 = \beta_2 = x - 3g + 2(h - y/3) = U + 2V, \;\; \beta_3 = x - 3g - 2(h - y/3) = U - 2V,$$

and thus are all odd (since U is odd and V is even), as required.

One can also check that

$$\begin{aligned} \frac{\alpha_k}{p} &= b\left(2v\left(h - \frac{y}{3}\right) + x\right) + p\chi\left(h - \frac{y}{3}\right) - p\frac{y}{3} - 2pu\frac{y}{3} \\ &= b\,(2vV + x) + p\chi V - p\frac{y}{3} - 2pu\frac{y}{3}. \end{aligned} \tag{4.16}$$

With the choice for v stated in the theorem, $b(2vV + x)$ is a multiple of p. It is clear from (4.11) that h is odd, and thus, from its definition above, that y is also odd, and so $py/3$ is an odd multiple of p. It can also be seen that $2pu\frac{y}{3}$ and $p\chi V$ are even multiples of p. Hence if x is even, then $(2vV + x)$ is an even multiple of p, and thus α_k/p is an odd multiple of p for all b, and so α_k is an odd multiple of $\alpha_{k^2} = p^2$ for all b. Likewise, it can be seen that α_k is an odd multiple of $\alpha_{k^2} = p^2$ only for even b in the case x is odd. As with the proof of Theorem 1.5, what has been shown is enough to guarantee that $F_1(q) = F_2(q) = F_3(q) = 0$, thus completing the proof of (i).

As with Theorem 1.5, the proof of part (ii) is a little more technical. First set $z = 3/2$ in Proposition 3.1 and its proof. Also as in the proof of part (ii) of Theorem 1.5, make the replacements

$$g \to h, \qquad h \to -g, \qquad x \to y, \qquad y \to -x,$$

to recover the values for p and j that appear in part (i).

With these changes, it also follows that $\beta_1 = \beta_2 = U - 3V$, and $\beta_3 = -U - 3V$, and so all are odd, as needed, and

$$\begin{aligned} \frac{\alpha_k}{p} &= b(v(x - 3g) + y) + \frac{p}{2}\chi(x - 3g) + pux + \frac{p}{2}x, \\ &= b(vU + y) + \frac{p}{2}\chi U + pux + \frac{p}{2}x. \end{aligned} \tag{4.17}$$

As previously, it may be seen from the second equation at (3.14) that $3|(p - j)$ (after making the substitution/transformations above). Since p has the form $24t + 1$, then $j \equiv 1 \pmod 3$. Hence from the first equation at (3.6), $3|(u + bv + w)$, or $u = 2w + 2bv + 3T$, for some integer T. Upon substituting for w from (1.9),

$$u = j + 3 + \chi p + 2bv + 3T, \tag{4.18}$$

$$\Longrightarrow \frac{\alpha_k}{p} = b(v(U + 2px) + y) + \frac{1}{2}(2j + 7)px + 3pTx + \frac{p}{2}\chi(U + 2px).$$

With the value stated for v in the theorem, $p|(vU+y)$. From the stated values for U and p,

$$2px+U \equiv (2x^3+x) = x(2x^2+1) \equiv 0 \pmod 3.$$

Thus $3p|(v(U+2px)+y)$ (recall $3|y$) and of course $3p|3pTx$. If x is odd, then j is odd and $\chi=0$, and since $j \equiv p \equiv 1 \pmod 3$, then $(2j+7)px/2$ is an odd multiple of $3p/2$. If x is even, then j is even and $\chi=1$, so that $(2j+7)px/2$ is a multiple of $3p$ and $p\chi(U+2px)/2$ is an odd multiple of $3p/2$. Hence α_k is an odd multiple of $3p^2/2=\alpha_{k^2}$ for all b, and the proof of (ii) is complete. □

Example 5. *Let $p=73=24(3)+1=(7^2)+6(2^2)$. So $(U,V)=(7,2)$. Let $(h,g)=(1,-1)$ be a solution to*

$$hU+3gV = h(7)+3g(2) = 1.$$

Then $x=U+3g=7+3(-1)=4$, and $y=3(h-V)=3(1-2)=-3$. Next, $j=xU-2yV=(4)(7)-2(-3)(2)=40$. Since j is even, $w=(j+3+p)/2=58$. Next $-x(2V)^{-1}=-(4)(4)^{-1}=-1\equiv 72 \pmod{73}$, so $v=72$. Hence, by the theorem, if the sequence $\{r_n\}$ is defined by

$$(q^b,q^{73-b};q^{73})_\infty^3(q^{40b},q^{146-40b};q^{146})_\infty = \sum_{n=0}^\infty r_nq^n,$$

then $r_{73n+72b^2+58b}=0$, for all integers n and b.

From part (ii) of Theorem 4.2, with $y=-3$ and $U=7$,

$$-y(U)^{-1} = 3(7)^{-1} \equiv 63 \pmod{73},$$

and $r_{73n+63b^2+58b}=0$, for all integers n and all integers b.

Finally, we have the following theorem for primes p of the form $p=24t+7$.

Theorem 4.3. *Let p be a prime of the form $p=24t+7$, so that $p=U^2+6V^2$ for integers U and V, both odd. Let h (odd) and g (even) be integers such that*

$$hU+3gV=1, \tag{4.19}$$

and set $x=3g+U$ and $y=3(h-V)$. Let

$$j = x(x-3g)-2y\left(h-\frac{y}{3}\right) = xU-2yV.$$

Let v and w $(0\le v,w\le p-1)$ be defined by

$$v \equiv -x(2V)^{-1} \pmod p, \tag{4.20}$$

$$w \equiv \frac{j+3}{2} \pmod p.$$

Let b be an odd integer and let the sequence $\{r_n\}$ be defined by

$$(q^b,q^{p-b};q^p)_\infty^3(q^{jb},q^{2p-jb};q^{2p})_\infty = \sum_{n=0}^\infty r_nq^n. \tag{4.21}$$

Then $r_{pn+vb^2+wb}=0$ *for all integers* n.

Proof. As with the proof of part (i) of Theorem 4.3, the proof here also follows from the $z=1$ case of Proposition 3.1. From (4.15) it may be seen that $\beta_1=\beta_2$ and β_3 are odd. Upon considering (4.16), α_k/p is given once again by

$$\frac{\alpha_k}{p}=b\,(2vV+x)+p\chi V-p\frac{y}{3}-2pu\frac{y}{3}, \tag{4.22}$$

except that j is odd (since x and U are odd) so $\chi=0$. It is also clear from the statement of the theorem that since x is odd, then y is even, and thus $py/3$ and $2puy/3$ are even multiples p. The conditions given for b, v and x give that $b\,(2vV+x)$ is an odd multiple of p. Hence α_k is an odd multiple of $p^2=\alpha_{k^2}$, completing the proof of Theorem 4.3.

□

Example 6. *Let* $p=103=24(4)+7=(7^2)+6(3^2)$. *So* $(U,V)=(7,3)$. *Let* $(h,g)=(-5,4)$ *be a solution to*

$$hU+3gV=h(7)+3g(3)=1.$$

Then $x=U+3g=7+3(4)=19$, *and* $y=3(h-V)=3(-5-3)=-24$. *Next,* $j=xU-2yV=(19)(7)-2(-24)(3)=277$. *Since* j *is odd,* $w\equiv(j+3)/2=140\equiv 37 \pmod{103}$. *Since* $x=19$ *and* $V=3$,

$$-x(2V)^{-1}=-19(6)^{-1}\equiv 14 \pmod{103},$$

so $v=14$. *Thus if the sequence* $\{r_n\}$ *is defined by*

$$(q^b,q^{103-b};q^{103})^3_\infty(q^{277b},q^{206-277b};q^{206})_\infty=\sum_{n=0}^{\infty}r_nq^n, \tag{4.23}$$

then $r_{103n+14b^2+37b}=0$, *for all integers* n *and all odd integers* b.

5. Concluding Remarks

There are a number of obvious questions that may be asked about vanishing coefficients in infinite products similar to

$$(q^{jb},q^{2p-jb},q^{2p};q^{2p})_\infty(q^b,q^{p-b},q^p;q^p)^3_\infty, \tag{5.1}$$

the topic of the present paper. Specifically, one could ask if there are there any vanishing coefficient results in any of the following situations:

1) products of the form (where the exponent 3 is switched to the first triple product)

$$(q^{jb},q^{2p-jb},q^{2p};q^{2p})^3_\infty(q^b,q^{p-b},q^p;q^p)_\infty; \tag{5.2}$$

2) products like (5.1) or (5.2), but with the first two powers of q in either or both of the triple product replaced with their negatives;

3) products like (5.1) or (5.2) or those described in 2), but with the $2p$ in the exponents replaced with kp, for some positive integer $k\neq 2$.

More generally, one could look at the product

$$(\pm q^{jb}, \pm q^{kp-jb}, q^{kp}; q^{kp})_\infty^r (\pm q^b, \pm q^{p-b}, q^p; q^p)_\infty^s, \tag{5.3}$$

where all of k, r and s are positive integers, and ask for a complete description of all the situations in which there are vanishing coefficient results.

We hope to provide at least partial answers to some of these questions in subsequent papers.

References

[1] Alladi, K.; Gordon B. *Vanishing coefficients in the expansion of products of Rogers-Ramanujan type.* Proc. Rademacher Centenary Conference, (G. E. Andrews and D. Bressoud, Eds.), Contemp. Math. **166**, (1994), 129–139.

[2] Andrews, G. E.; Bressoud, D. M. *Vanishing coefficients in infinite product expansions.* J. Austral. Math. Soc. Ser. A **27** (1979), no. 2, 199-202.

[3] Baruah, N.D., Kaur, M. *Some results on vanishing coefficients in infinite product expansions.* Ramanujan J. **53** (2020), no. 3, 551–568.

[4] Cao, Z. *Integer Matrix Exact Covering Systems and Product Identities for Theta Functions.* Int. Math. Res. Not. no. **19** (2011), 44714514.

[5] Hirschhorn, M. D. *Two remarkable q-series expansions.* Ramanujan J. **49**(2019), no. 2, pp 451–463.

[6] Mc Laughlin, J., *A Generalization of Schröter's Formula.* Ann. Comb. **23** (2019), no. 3–4, 889–906.

[7] Mc Laughlin, J., *Further results on vanishing coefficients in infinite product expansions.* J. Aust. Math. Soc. **98** (2015), no. 1, 69–77.

[8] Mc Laughlin, J. *New infinite q-product expansions with vanishing coefficients.* Ramanujan J **55**, 733–760 (2021). https://doi.org/10.1007/s11139-020-00275-w

[9] Richmond, B.; Szekeres, G. *The Taylor coefficients of certain infinite products.* Acta Sci. Math. (Szeged) **40** (1978), no. 3–4, 347-369.

[10] Tang, D. *Vanishing coefficients in some q-series expansions.* Int. J. Number Theory **15** (2019), no. 4, 763–773.

Mathematics Department, 25 University Avenue, West Chester University, West Chester, PA 19383

E-mail address: jmclaughlin2@wcupa.edu

Mathematics Department, 25 University Avenue, West Chester University, West Chester, PA 19383

E-mail address: pzimmer@wcupa.edu