SOME NEW FAMILIES OF TASOEVIAN- AND HURWITZIAN CONTINUED FRACTIONS

JAMES MC LAUGHLIN

ABSTRACT. We derive closed-form expressions for several new classes of Hurwitzian- and Tasoevian continued fractions, including

$$[0; \overline{p-1, 1, u(a+2nb)-1, p-1, 1, v(a+(2n+1)b)-1}\,]_{n=0}^{\infty},$$

$[0; \overline{c+dm^n}]_{n=1}^{\infty}$ and $[0; \overline{eu^n, fv^n}]_{n=1}^{\infty}$. One of the constructions used to produce some of these continued fractions can be iterated to produce both Hurwitzian- and Tasoevian continued fractions of arbitrary long quasi-period, with arbitrarily many free parameters and whose limits can be determined as ratios of certain infinite series.

We also derive expressions for arbitrarily long *finite* continued fractions whose partial quotients lie in arithmetic progressions.

1. INTRODUCTION

In this paper we exhibit several new infinite families of regular continued fraction of *Hurwitzian*- and *Tasoevian* type, continued fractions whose value can expressed in terms of certain infinite series.

Hurwitzian continued fractions ([2], [3]) are of the form

$$[a_0; a_1, \cdots, a_k, f_1(1), \cdots, f_n(1), f_1(2), \cdots, f_n(2), \cdots]$$
$$=: [a_0; a_1, \cdots, a_k, \overline{f_1(m), \cdots, f_n(m)}\,]_{m=1}^{\infty}.$$

Here the $f_i(x)$ are polynomials with rational coefficients taking only positive integral values for integral $x \geq 1$ and at least one is non-constant. The integer n is termed the *quasi-period* of the continued fraction. The closed form for Hurwitzian continued fractions is not known in general. This class contains numbers like

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \dots] = [2; \overline{1, 2m, 1}\,]_{m=1}^{\infty},$$
$$\tan 1 = [1; 1, 1, 3, 1, 5, 1, 7, 1, 9, \dots] = [1; \overline{2m-1, 1}]_{m=1}^{\infty}$$

These continued fractions were also investigated by D. N. Lehmer [11] and more recently by Komatsu in [5], [7] and [8]. A nice example that follows

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from Lambert's continued fraction [10]

$$\frac{e^z - e^{-z}}{e^z - e^{-z}} = \frac{z}{1+}\ \frac{z^2}{3+}\ \frac{z^2}{5+}\ \frac{z^2}{7+}\cdots \tag{1.1}$$

is the following (see also [5]):

$$\sqrt{\frac{v}{u}}\tan\frac{1}{\sqrt{uv}} = [0; u-1, \overline{1, (4k-1)v-2, 1, (4k+1)u-2}]_{k=1}^{\infty}.$$

A a sub-class of Hurwitzian continued fractions (with all polynomials $f_i(x)$ of degree 1) is due to D.H. Lehmer [12], who found closed forms for the numbers represented by regular continued fractions whose partial quotients were terms in an arithmetic progression,

$$[0; a, a+b, a+2b, a+3b, \cdots] = \frac{I_{(a/b)}(2/b)}{I_{(a/b)-1}(2/b)}, \tag{1.2}$$

where

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2m}}{\Gamma(m+1)\Gamma(\nu+m+1)}.$$

More transparently,

$$[0; a, a+b, a+2b, a+3b, \cdots] = \frac{1}{b}\frac{\displaystyle\sum_{k=0}^{\infty}\frac{b^{-2k}}{(a/b)_{k+1}k!}}{\displaystyle\sum_{k=0}^{\infty}\frac{b^{-2k}}{(a/b)_k k!}},$$

where $(a/b)_0 = 1$ and $(a/b)_k = (a/b)(a/b+1)\ldots(a/b+k-1)$ for $k > 0$.

Lehmer also evaluated continued fractions whose partial quotients consisted of two interlaced arithmetic progressions. Let a, b, c and d be integers satisfying

$$2bc = d(2a+b).$$

Then

$$[0; a, c, a+b, c+d, a+2b, c+2d, \cdots] = \sqrt{\frac{d}{b}}\,\frac{I_{(2a/b)}(4/\sqrt{bd})}{I_{(2a/b)-1}(4/\sqrt{bd})}.$$

An example that Lehmer gave of the former type was the following:

$$[1; 2, 3, 4, 5, \cdots] = \frac{\sum_{m=0}^{\infty}\frac{1}{(m!)^2}}{\sum_{m=0}^{\infty}\frac{1}{m!(m+1)!}}$$

Tasoev [15], [16] proposed a new type of continued fraction of the form

$$[a_0; \underbrace{a, \cdots, a}_{m}, \underbrace{a^2, \cdots, a^2}_{m}, \underbrace{a^3, \cdots, a^3}_{m}, \cdots], \tag{1.3}$$

where $a_0 \geq 0$, $a \geq 2$ and $m \geq 1$ are integers. This type was further investigated by Komatsu in [4], where he derived a closed form for the general case ($m \geq 1$, arbitrary). Komatsu gave several variations of Tasoevian continued fractions in [5], [6], [7] and [8]. In [14], the present author and Nancy Wyshinski derived several variations of Tasoev's continued fraction from known results about q-continued fractions. Two examples of our results from that paper are the following.

Example 1. *Define*

$$F(c,d,q) := \sum_{n=0}^{\infty} \frac{(-1)^n c^n q^{n(n+1)/2}}{(q;q)_n (cq/d;q)_n}$$

and let $\omega = e^{2\pi i/3}$. *If* $a > 1$ *is an integer and* c *is a rational such that* a/c *is an integer,* $a/c > 2$, *then*

$$\left[0; \frac{a}{c} - 2, \overline{1, \frac{a^{k+1}}{c} - 3}\right]_{k=1}^{\infty} = \frac{c/a\, F(-c\omega/a, \omega^2, 1/a)}{(1 + c\omega^2/a) F(-c\omega, \omega^2, 1/a)}.$$

Example 2. *For* r, s *and* $q \in \mathbb{C}$ *with* $|q| < 1$, *define*

$$\phi(r,s,q) = \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2} r^n}{(q;q)_n (-sq;q)_n}.$$

Let m *and* n *be positive integers and let* d *be rational such that* $dn \in \mathbb{Z}^+$ *and* $dmn > 1$. *If* $n > 2$ *and* $m > 1$ *then*

$$[0, \overline{1, d^{2k-2}n^{2k-1} - 2, 1, m^{2k-1} - 1, d^{2k-1}n^{2k}, m^{2k} - 1}]_{k=1}^{\infty}$$
$$= 1 + \frac{\phi(dm, d, -1/(dmn))}{\phi(-1/n, d, -1/(dmn))}.$$

In the present paper we continue our work with q-continued fractions, giving q-continued fraction proofs for some existing families of Tasoevian and Hurwitzian continued fractions. In addition, we also find the limits of some new families of Tasoevian and Hurwitzian continued fractions.

We also evaluate various finite continued fractions containing arithmetic progressions, deriving Lehmer's results in the limit.

2. Tasoevian Continued Fractions

In [1], the following result on q-continued fractions was proved.

Theorem 1. *Let* a, b, c, d *be complex numbers with* $d \neq 0$ *and* $|q| < 1$. *Define*

$$H_1(a,b,c,d,q) := \frac{1}{1} + \frac{-abq + c}{(a+b)q + d} + \cdots + \frac{-abq^{2n+1} + cq^n}{(a+b)q^{n+1} + d} + \cdots.$$

Then

$$\frac{1}{H_1(a,b,c,d,q)} - 1 = \frac{c - abq}{(d+aq)q} \frac{\sum_{j=0}^{\infty} \dfrac{(b/d)^j (-c/bd)_j\, q^{(j+1)(j+2)/2}}{(q)_j (-aq^2/d)_j}}{\sum_{j=0}^{\infty} \dfrac{(b/d)^j (-c/bd)_j\, q^{j(j+1)/2}}{(q)_j (-aq/d)_j}}. \tag{2.1}$$

Here we are employing the standard notation for q-products:

$$(z)_0 := (z;q)_0 := 1, \quad (z)_n := (z;q)_n := \prod_{k=0}^{n-1}(1 - z\,q^k), \quad \text{if } n \geq 1.$$

This theorem immediately leads to some general results concerning Tasoevian continued fractions.

Theorem 2. *Let c, e and m be integers, $m > 1$ and let d be a rational such that $dm^2, dcm/e \in \mathbb{N}$, and $c + dcm/e, e + dm^2 > 0$. Let*

$$a = \frac{e - \sqrt{e^2 + 4e/c}}{2}, \qquad b = \frac{e + \sqrt{e^2 + 4e/c}}{2}.$$

Then

$$\left[0; \overline{c + \frac{dc}{e}m^{2n-1}, e + dm^{2n}}\,\right]_{n=1}^{\infty} \tag{2.2}$$

$$= \frac{e/c}{md + a} \frac{\sum_{n=0}^{\infty} \dfrac{(b/d)^n m^{-n(n+3)/2}}{(1/m;1/m)_n(-a/dm^2;1/m)_n}}{\sum_{n=0}^{\infty} \dfrac{(b/d)^n m^{-n(n+1)/2}}{(1/m;1/m)_n(-a/dm;1/m)_n}}.$$

Proof. With the stated values of a and b,

$$\begin{aligned}
&[0;\overline{c + \frac{dc}{e}m^{2n-1}, e + dm^{2n}}]_{n=1}^{\infty} \\
&= \frac{1}{-((a+b)+dm)/(ab)} + \frac{1}{(a+b)+dm^2} + \\
&\qquad \cdots + \frac{1}{-((a+b)+dm^{2n-1})/(ab)} + \frac{1}{(a+b)+dm^{2n}} + \cdots \\
&= \frac{-ab}{(a+b)+dm} + \frac{-ab}{(a+b)+dm^2} + \cdots + \frac{-ab}{(a+b)+dm^n} + \cdots \\
&= \frac{-ab/m}{(a+b)/m+d} + \frac{-ab/m^3}{(a+b)/m^2+d} + \cdots + \frac{-ab/m^{2n-1}}{(a+b)/m^n+d} + \cdots.
\end{aligned}$$

The result now follows from (2.1), upon setting $q = 1/m$ and $c = 0$. □

Corollary 1. *Let c and m be integers, $m > 1$ and let d be a positive rational such that $dm \in \mathbb{N}$ and $c + dm > 0$. Let*

$$a = \frac{c - \sqrt{c^2 + 4}}{2}, \qquad b = \frac{c + \sqrt{c^2 + 4}}{2}.$$

Then

$$[0; \overline{c + dm^n}]_{n=1}^{\infty} = \frac{1}{md + a} \frac{\sum_{n=0}^{\infty} \dfrac{(b/d)^n m^{-n(n+3)/2}}{(1/m;1/m)_n(-a/dm^2;1/m)_n}}{\sum_{n=0}^{\infty} \dfrac{(b/d)^n m^{-n(n+1)/2}}{(1/m;1/m)_n(-a/dm;1/m)_n}}. \tag{2.3}$$

Proof. Let $e = c$ in Theorem 2.2. □

Remarks: 1) We believe that the limit of the general Tasoevian continued fraction of the form $[0; \overline{c + dm^n}]_{n=1}^{\infty}$ has not been evaluated before, although special cases have occurred in the literature, such as $[0; \overline{dm^n}]_{n=1}^{\infty}$ by Komatsu in [5]. We believe that the evaluation of the general Tasoevian continued fraction $[0; \overline{c + \frac{dc}{e} m^{2n-1}, e + dm^{2n}}]_{n=1}^{\infty}$ is also new.

2) It is clear from Theorem 1 that (2.3) also holds for many cases where the partial quotients in (2.2) or (2.3) are not positive integers. In particular, we can let the parameters assume negative values and then convert the resulting continued fractions to regular continued fractions by removing any resulting zero- and negative partial quotients. This will produce still further general classes of Tasoevian continued fractions.

To accomplish this, we recall, as noted in [17], that $[m, n, 0, p, \alpha] = [m, n+p, \alpha]$ and $[m, -n, \alpha] = [m-1, 1, n-1, -\alpha]$. We give two examples to illustrate the phenomenon, using the continued fraction at (2.2)

Corollary 2. *Let c, e and m be integers, $m > 1$ and let d be a positive rational such that $dm^2, dcm/e \in \mathbb{N}$. Let*

$$a = \frac{e - \sqrt{e^2 + 4e/c}}{2}, \qquad b = \frac{e + \sqrt{e^2 + 4e/c}}{2}.$$

(i) Suppose that $dcm/e - c - 2 > 0$ and $dm^2 + e - 2 > 0$. Then

$$\left[0; \overline{1, \frac{dc}{e} m^{2n-1} - c - 2, 1, dm^{2n} + e - 2}\right]_{n=1}^{\infty} = 1 + \frac{e/c}{-md + a} \frac{\sum_{n=0}^{\infty} \frac{(b/d)^n (-m)^{-n(n+3)/2}}{(-1/m; -1/m)_n (-a/dm^2; -1/m)_n}}{\sum_{n=0}^{\infty} \frac{(b/d)^n (-m)^{-n(n+1)/2}}{(-1/m; -1/m)_n (a/dm; -1/m)_n}}. \tag{2.4}$$

(ii) Suppose that $dcm/e + c - 1 > 0$ and $dm^2 - e - 2 > 0$. Then

$$\left[0; \frac{dc}{e} m + c - 1, \overline{1, dm^{2n} - e - 2, 1, \frac{dc}{e} m^{2n+1} + c - 2}\right]_{n=1}^{\infty} = \frac{e/c}{md + a} \frac{\sum_{n=0}^{\infty} \frac{(-b/d)^n (-m)^{-n(n+3)/2}}{(-1/m; -1/m)_n (a/dm^2; -1/m)_n}}{\sum_{n=0}^{\infty} \frac{(-b/d)^n (-m)^{-n(n+1)/2}}{(-1/m; -1/m)_n (-a/dm; -1/m)_n}}. \tag{2.5}$$

Proof. The identity at (2.4) follows from (2.2) upon replacing m by $-m$, removing the negative partial quotients from the continued fraction as described above, and finally moving the initial -1 to the right side. The identity at (2.4) follows similarly, upon replacing d by $-d$ and m by $-m$. □

Before coming to the next result, we need some more terminology. We call $d_0 + K_{n=1}^{\infty} c_n/d_n$ a *canonical contraction* of $b_0 + K_{n=1}^{\infty} a_n/b_n$ if

$$C_k = A_{n_k}, \qquad D_k = B_{n_k} \qquad \text{for } k = 0, 1, 2, 3, \dots, \tag{2.6}$$

where C_n, D_n, A_n and B_n are canonical numerators and denominators of $d_0 + K_{n=1}^{\infty} c_n/d_n$ and $b_0 + K_{n=1}^{\infty} a_n/b_n$ respectively. From [13] (page 83) we have the following theorem:

Theorem 3. *The canonical contraction of* $b_0 + K_{n=1}^{\infty} a_n/b_n$ *with*

$$C_k = A_{2k} \qquad D_k = B_{2k} \qquad \text{for } k = 0, 1, 2, 3, \dots,$$

exists if and only if $b_{2k} \neq 0$ *for* $k = 1, 2, 3, \dots$, *and in this case is given by*

$$b_0 + \frac{b_2 a_1}{b_2 b_1 + a_2} - \frac{a_2 a_3 b_4/b_2}{a_4 + b_3 b_4 + a_3 b_4/b_2} - \frac{a_4 a_5 b_6/b_4}{a_6 + b_5 b_6 + a_5 b_6/b_4} + \cdots. \tag{2.7}$$

The continued fraction (2.7) is called the *even* part of $b_0 + K_{n=1}^{\infty} a_n/b_n$. If a continued fraction converges then of course its even part converges to the same limit.

Theorem 4. *Let* u, *and* v *be positive integers,* $u, v > 1$, *and let* e *and* f *be rationals such that* $eu, fv \in \mathbb{N}$. *Then*

$$[0; \overline{eu^n, fv^n}]_{n=1}^{\infty} = \left(\frac{1}{eu} - \frac{1}{e^2 f u^2 v + e}\right) \frac{\sum_{n=0}^{\infty} \frac{(ef)^{-n} (uv)^{-n(n+3)/2}}{(1/uv; 1/uv)_n (-1/efu^3 v^2; 1/uv)_n}}{\sum_{n=0}^{\infty} \frac{(ef)^{-n} (uv)^{-n(n+1)/2}}{(1/uv; 1/uv)_n (-1/efu^2 v; 1/uv)_n}}. \tag{2.8}$$

Proof. We consider the continued fraction

$$[0; b_1, \overline{eu^n, fv^n}]_{n=1}^{\infty}, \tag{2.9}$$

with b_1 an arbitrary positive integer. Clearly this continued fraction converges, and is thus equal to its even part. By (2.7) this equals

$$\begin{aligned} &\frac{eu}{eub_1 + 1} - \frac{u}{1 + (efu)(uv) + u} - \frac{u}{1 + (efu)(uv)^2 + u} - \cdots \\ &= \frac{eu}{eub_1 + 1} - \frac{u/(uv)}{(1+u)/(uv) + efu} - \frac{u/(uv)^3}{(1+u)/(uv)^2 + efu} - \cdots. \end{aligned} \tag{2.10}$$

We now apply Theorem 1 to the first tail of the continued fraction above, setting $q = 1/uv$, $c = 0$, $d = efu$, $a = 1$ and $b = u$. The result follows upon inverting both the expression resulting from (2.10) and the continued fraction at (2.9), and then cancelling b_1. □

Remark: Komatsu has a result in [5], concerning Tasoevian continued fractions of the form $[0; \overline{ua^k, vb^k}]_{k=1}^{\infty}$, but he does not explicitly compute the limits, expressing them instead as ratios of series containing certain functions, $R_{0,n}$ and $R_{1,n}$, which are defined recursively for $n \geq 0$. We believe the result in Theorem 4 to be new.

In [12], where Lehmer investigated continued fractions whose partial quotients were in arithmetical progressions, he remarked that it was also possible to evaluate continued fractions in which the terms forming the arithmetic progressions were separated by constant strings of arbitrary partial quotients. We next show that this can also be done with some classes of Tasoevian continued fractions.

Theorem 5. *Let c, e and m be integers, with $m > 1$.*

Let $a_1, a_2, \ldots, a_k$ be fixed positive integers and, for $1 \leq i \leq k$, define P_i and Q_i by

$$\frac{P_i}{Q_i} = \frac{1}{a_1 +}\,\frac{1}{a_2 +}\cdots\frac{1}{a_i},$$

and set $C = Q_{k-1} + P_k + c\,Q_k$ and $E = Q_{k-1} + P_k + e\,Q_k$. We suppose further that d is a positive rational such that $Cd\,m/E$, $dm \in \mathbb{N}$.

If k is even, set

$$a = \frac{E - \sqrt{E^2 + 4E/C}}{2},$$
$$b = \frac{E + \sqrt{E^2 + 4E/C}}{2}.$$

Then

$$\left[0; \overline{a_1, \ldots, a_k, c + \frac{C}{E}dm^{2n-1}, a_1, \ldots, a_k, e + dm^{2n}}\,\right]_{n=1}^{\infty} \tag{2.11}$$
$$= \frac{P_k}{Q_k} + \frac{E/C}{Q_k(mdQ_k + a)} \frac{\sum_{n=0}^{\infty} \dfrac{(b/dQ_k)^n m^{-n(n+3)/2}}{(1/m;1/m)_n(-a/dQ_k m^2;1/m)_n}}{\sum_{n=0}^{\infty} \dfrac{(b/dQ_k)^n m^{-n(n+1)/2}}{(1/m;1/m)_n(-a/dQ_k m;1/m)_n}}.$$

If k is odd, set

$$a = \frac{E - \sqrt{E^2 - 4E/C}}{2},$$
$$b = \frac{E + \sqrt{E^2 - 4E/C}}{2}.$$

Then

$$\left[0; \overline{a_1, \ldots, a_k, c + \frac{C}{E}dm^{2n-1}, a_1, \ldots, a_k, e + dm^{2n}}\,\right]_{n=1}^{\infty} \tag{2.12}$$
$$= \frac{P_k}{Q_k} + \frac{-E/C}{Q_k(mdQ_k + a)} \frac{\sum_{n=0}^{\infty} \dfrac{(b/dQ_k)^n m^{-n(n+3)/2}}{(1/m;1/m)_n(-a/dQ_k m^2;1/m)_n}}{\sum_{n=0}^{\infty} \dfrac{(b/dQ_k)^n m^{-n(n+1)/2}}{(1/m;1/m)_n(-a/dQ_k m;1/m)_n}}.$$

Proof. For any α,

$$\begin{aligned}[0; a_1, a_2, \dots a_k, \alpha] &= \frac{\alpha P_k + P_{k-1}}{\alpha Q_k + Q_{k-1}}\\ &= \frac{P_k}{Q_k} + \frac{(P_{k-1}Q_k - Q_{k-1}P_k)/Q_k^2}{Q_{k-1}/Q_k + \alpha}\\ &= \frac{P_k}{Q_k} + \frac{(-1)^k/Q_k^2}{Q_{k-1}/Q_k + \alpha},\end{aligned}$$

where the last equality follows from a standard identity in continued fractions. Thus

$$\begin{aligned}&\left[0; \overline{a_1, \dots, a_k, c + \frac{C}{E} dm^{2n-1}, a_1, \dots, a_k, e + dm^{2n}}\right]_{n=1}^{\infty}\\ &= \frac{P_k}{Q_k} + \frac{(-1)^k/Q_k^2}{\frac{P_k+Q_{k-1}}{Q_k} + c + \frac{dCm}{E}} + \frac{(-1)^k/Q_k^2}{\frac{P_k+Q_{k-1}}{Q_k} + e + dm^2} + \cdots\\ &= \frac{P_k}{Q_k} + \frac{1}{Q_k}\left(\frac{(-1)^k}{C + \frac{CdQ_k}{E}m} + \frac{(-1)^k}{E + dQ_k m^2} + \cdots\right).\end{aligned}$$

If k is even, then

$$\begin{aligned}&\left[0; \overline{a_1, \dots, a_k, c + \frac{C}{E} dm^{2n-1}, a_1, \dots, a_k, e + dm^{2n}}\right]_{n=1}^{\infty}\\ &\qquad = \frac{P_k}{Q_k} + \frac{1}{Q_k}\left[0; \overline{C + \frac{CdQ_k}{E} m^{2n-1}, E + dQ_k m^{2n}}\right]_{n=1}^{\infty},\end{aligned}$$

and (2.11) now follows from Theorem 2.

If k is odd, then

$$\begin{aligned}&\left[0; \overline{a_1, \dots, a_k, c + \frac{C}{E} dm^{2n-1}, a_1, \dots, a_k, e + dm^{2n}}\right]_{n=1}^{\infty}\\ &\qquad = \frac{P_k}{Q_k} + \frac{1}{Q_k}\left[0; \overline{(-C) + \frac{(-C)dQ_k}{E} m^{2n-1}, E + dQ_k m^{2n}}\right]_{n=1}^{\infty},\end{aligned}$$

and (2.12) likewise follows from Theorem 2. □

Corollary 3. *Let c and m be integers, $m > 1$ and let d be a positive rational such that $dm \in \mathbb{N}$ and $c + dm > 0$.*

Let $a_1, a_2, \dots, a_k$ be fixed positive integers and, for $1 \le i \le k$, define P_i, Q_i by

$$\frac{P_i}{Q_i} = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_i},$$

and set $C = Q_{k-1} + P_k + c\,Q_k$.

If k is even, set

$$a = \frac{C - \sqrt{C^2 + 4}}{2},$$
$$b = \frac{C + \sqrt{C^2 + 4}}{2}.$$

Then

$$[0; \overline{a_1, a_2, \dots a_k, c + dm^n}]_{n=1}^{\infty} \tag{2.13}$$
$$= \frac{P_k}{Q_k} + \frac{1}{Q_k(mdQ_k + a)} \frac{\sum_{n=0}^{\infty} \frac{(b/dQ_k)^n m^{-n(n+3)/2}}{(1/m; 1/m)_n(-a/dQ_k m^2; 1/m)_n}}{\sum_{n=0}^{\infty} \frac{(b/dQ_k)^n m^{-n(n+1)/2}}{(1/m; 1/m)_n(-a/dQ_k m; 1/m)_n}}.$$

If k is odd, set

$$a = \frac{C - \sqrt{C^2 - 4}}{2},$$
$$b = \frac{C + \sqrt{C^2 - 4}}{2}.$$

Then

$$[0; \overline{a_1, a_2, \dots a_k, c + dm^n}]_{n=1}^{\infty} \tag{2.14}$$
$$= \frac{P_k}{Q_k} + \frac{-1}{Q_k(mdQ_k + a)} \frac{\sum_{n=0}^{\infty} \frac{(b/dQ_k)^n m^{-n(n+3)/2}}{(1/m; 1/m)_n(-a/dQ_k m^2; 1/m)_n}}{\sum_{n=0}^{\infty} \frac{(b/dQ_k)^n m^{-n(n+1)/2}}{(1/m; 1/m)_n(-a/dQ_k m; 1/m)_n}}.$$

Proof. Set $e = c$ in Theorem 5. □

Corollary 4. *Let c and m be integers, $m > 1$ and let d be a positive rational such that $dm \in \mathbb{N}$ and $c + dm > 0$. Let k be an even positive integer, let F_i denote the i-th Fibonacci number and set $C = 2F_k + c\,F_{k+1}$. Set*

$$a = \frac{C - \sqrt{C^2 + 4}}{2},$$
$$b = \frac{C + \sqrt{C^2 + 4}}{2}.$$

Then

$$[0; \overline{\underbrace{1, 1, \dots, 1, 1}_{k}, c + dm^n}]_{n=1}^{\infty} \tag{2.15}$$

$$= \frac{F_k}{F_{k+1}} + \frac{1}{F_{k+1}(mdF_{k+1} + a)} \frac{\sum_{n=0}^{\infty} \frac{(b/dF_{k+1})^n m^{-n(n+3)/2}}{(1/m; 1/m)_n(-a/dF_{k+1} m^2; 1/m)_n}}{\sum_{n=0}^{\infty} \frac{(b/dF_{k+1})^n m^{-n(n+1)/2}}{(1/m; 1/m)_n(-a/dF_{k+1} m; 1/m)_n}}.$$

Proof. This follows immediately from Corollary 3, upon noting that

$$[0; \underbrace{1, 1, \ldots, 1, 1}_{i}] = \frac{F_i}{F_{i+1}}.$$

□

We also require some preliminary results before our next construction (see also (2.6) above). The following theorem can be found in [13], page 85.

Theorem 6. *The canonical contraction of* $b_0 + K_{n=1}^{\infty} a_n/b_n$ *with* $C_0 = A_1/B_1$

$$C_k = A_{2k+1} \qquad D_k = B_{2k+1} \qquad \text{for } k = 1, 2, 3, \ldots ,$$

exists if and only if $b_{2k+1} \neq 0 for K = 0, 1, 2, 3, \ldots$, *and in this case is given by*

$$(2.16) \quad \frac{b_0 b_1 + a_1}{b_1} - \frac{a_1 a_2 b_3/b_1}{b_1(a_3 + b_2 b_3) + a_2 b_3} - \frac{a_3 a_4 b_5 b_1/b_3}{a_5 + b_4 b_5 + a_4 b_5/b_3} - \frac{a_5 a_6 b_7/b_5}{a_7 + b_6 b_7 + a_6 b_7/b_5} - \frac{a_7 a_8 b_9/b_7}{a_9 + b_8 b_9 + a_8 b_9/b_7} + \cdots .$$

The continued fraction (2.16) is called the *odd* part of $b_0 + K_{n=1}^{\infty} a_n/b_n$. The following corollary follows easily from Theorem 6.

Corollary 5. *The odd part of the continued fraction*

$$\frac{c_1}{1} - \frac{c_2}{1} + \frac{c_2}{1} - \frac{c_3}{1} + \frac{c_3}{1} - \frac{c_4}{1} + \frac{c_4}{1} - \cdots$$

is

$$c_1 + \frac{c_1 c_2}{1} + \frac{c_2 c_3}{1} + \frac{c_3 c_4}{1} + \cdots .$$

This corollary implies the following result.

Corollary 6. *Let* p *and* b_i, $i \geq 1$ *be complex numbers. If the continued fraction*

$$(2.17) \quad \frac{1/p}{1} - \frac{p/b_1}{1} + \frac{p/b_1}{1} - \frac{1/pb_2}{1} + \frac{1/pb_2}{1} - \frac{p/b_3}{1} + \frac{p/b_3}{1} - \frac{1/pb_4}{1} + \frac{1/pb_4}{1} - \cdots - \frac{p/b_{2n-1}}{1} + \frac{p/b_{2n-1}}{1} - \frac{1/pb_{2n}}{1} + \frac{1/pb_{2n}}{1} - \cdots$$

converges, then

$$(2.18) \quad \left[0; \overline{p, \frac{-b_{2n-1}}{p^2}, -p, b_{2n}}\right]_{n=1}^{\infty} = \frac{1}{p} + [0; b_1, b_2, b_3, \ldots].$$

Proof. The continued fraction at (2.17) is easily seen to be equivalent to the continued fraction on the left side of (2.18), after a sequence of similarity transformations is applied to the former continued fraction to transform all the partial numerators into "1"'s. On the other hand, since the continued fraction at (2.17) converges, it is equal to its odd part, which, by Corollary 5, is the continued fraction

$$\begin{aligned}&\frac{1}{p}+\frac{1/b_1}{1}+\frac{1/b_1b_2}{1}+\frac{1/b_2b_3}{1}+\frac{1/b_3b_4}{1}+\cdots\\&=\frac{1}{p}+[0;b_1,b_2,b_3,\dots].\end{aligned}$$

□

We will also make use of Worpitzky's Theorem (see [13], pp. 35–36) to ensure convergence of the continued fraction at (2.17).

Theorem 7. *(Worpitzky) Let the continued fraction $K_{n=1}^{\infty}a_n/1$ be such that $|a_n|\leq 1/4$ for $n\geq 1$. Then$K_{n=1}^{\infty}a_n/1$ converges. All approximants of the continued fraction lie in the disc $|w|<1/2$ and the value of the continued fraction is in the disk $|w|\leq 1/2$.*

Corollary 6 can now be used to derive the limit of new Tasoevian continued fractions from existing Tasoevian continued fractions whose values are known. The new continued fraction will contain an additional free parameter. We give two examples.

Theorem 8. *Let c, e, $m>1$ and $p>1$ be integers. Let d be a positive rational such that $dm, dcm/e\in\mathbb{N}$, and $c+dcm/e-1, e+dm^2-1>0$. Let*

$$a=\frac{e-\sqrt{e^2+4e/cp^2}}{2},\qquad b=\frac{e+\sqrt{e^2+4e/cp^2}}{2}.$$

Then

$$\left[0;\overline{p-1,1,c+\frac{dc}{e}m^{2n-1}-1,p-1,1,e+dm^{2n}-1}\right]_{n=1}^{\infty}$$
$$=\frac{1}{p}+\frac{e/cp^2}{md+a}\frac{\sum_{n=0}^{\infty}\dfrac{(b/d)^nm^{-n(n+3)/2}}{(1/m;1/m)_n(-a/dm^2;1/m)_n}}{\sum_{n=0}^{\infty}\dfrac{(b/d)^nm^{-n(n+1)/2}}{(1/m;1/m)_n(-a/dm;1/m)_n}}. \tag{2.19}$$

Proof. Replace c by cp^2 in Theorem 2 and let the resulting continued fraction be the continued fraction on the right side of (2.18). After the negatives are removed (see the remark before Corollary 2) from the corresponding continued fraction on the left side of (2.18), the continued fraction on the left side at (2.19) is produced and the result follows. □

Theorem 9. *Let u, and v be positive integers, $u, v > 1$, and let e and f be rationals such that $eu - 1, fv - 1 \in \mathbb{N}$. Then*

$$[0; \overline{p-1, 1, eu^n - 1, p-1, 1, fv^n - 1}]_{n=1}^{\infty} = \frac{1}{p} + \left(\frac{1}{ep^2 u} - \frac{1}{e^2 f p^4 u^2 v + ep^2}\right) \frac{\sum_{n=0}^{\infty} \frac{(efp^2)^{-n} (uv)^{-n(n+3)/2}}{(1/uv; 1/uv)_n (-1/efp^2 u^3 v^2; 1/uv)_n}}{\sum_{n=0}^{\infty} \frac{(efp^2)^{-n} (uv)^{-n(n+1)/2}}{(1/uv; 1/uv)_n (-1/efp^2 u^2 v; 1/uv)_n}}. \tag{2.20}$$

Proof. The proof is similar to that of Theorem 8, except we replace e with ep^2 in Theorem 4. □

Remark: It is clear that many other continued fractions of Tasoevian type could be produced from those listed in this section, by either replacing various parameters by their negatives, or applying Corollary 6 differently (for example, by replacing m by mp or mp^2 (instead of replacing e by ep^2) in Theorems 8 and 9). However, we feel these methods have been sufficiently illustrated here and refrain from further examples.

3. Hurwitzian Continued Fractions

We first recall some of the well-known classes of Hurwitzian continued fractions and consider some elementary generalizations of them. We first note that Lehmer's continued fraction (1.2)

$$[0; a, a+b, a+2b, a+3b, \cdots] = \frac{I_{(a/b)}(2/b)}{I_{(a/b)-1}(2/b)},$$

can easily be generalized. Replace a by $a\sqrt{uv}$ and b by $b\sqrt{uv}$, multiply both sides of (1.2) by $\sqrt{v/u}$, apply a sequence of similarity transformations to the resulting continued fraction to make it regular once more, and we get

$$[0; ua, v(a+b), u(a+2b), v(a+3b), \cdots] = \sqrt{\frac{v}{u}} \frac{I_{(a/b)}(2/b\sqrt{uv})}{I_{(a/b)-1}(2/b\sqrt{uv})}. \tag{3.1}$$

Komatsu also derives this generalization in [5], but his derivation is more complicated. We note that several of the well-known classes of Hurwitzian continued fractions follow as special cases of (3.1). For example, before removing the negative partial quotients, Lambert's continued fraction (1.1) gives that

$$\sqrt{\frac{v}{u}} \tan \frac{1}{\sqrt{uv}} = [0, u, -3v, 5u, -7v, \ldots], \tag{3.2}$$

which follows upon setting $a = 1$, $b = 2$ and replacing v with $-v$. The continued fraction

$$\sqrt{\frac{v}{u}}\tanh\frac{1}{\sqrt{uv}} = [0, u, 3v, 5u, 7v, \dots], \tag{3.3}$$

is clearly also a special case. The continued fraction

$$[0; \overline{(4n+2)s}\,]_{n=0}^{\infty} = \frac{e^{1/s}-1}{e^{1/s}+1}$$

is clearly a special case of (1.2). Thus, as Komatsu indicated in [9], the continued fraction at (3.1) may be used to generalize several of the well-known Hurwitzian continued fraction expansions. As in Corollary 2, further variations follow upon replacing some of the parameters by their negatives.

We also recall a well-known continued fraction expansion for e^z, $z \in \mathbb{C}$ (see [13, page 563], for example):

$$e^z = \frac{1}{1-}\,\frac{z}{1+}\,\frac{z}{2-}\,\frac{z}{3+}\,\frac{z}{2-}\,\frac{z}{5+}\,\frac{z}{2-}\,\frac{z}{7+}\cdots. \tag{3.4}$$

Set $z = 1/m^2$ and apply a sequence of similarity transformations to the resulting continued fraction some to get that

$$\begin{aligned} m(1-e^{-1/m^2}) &= [0; \overline{(4n+1)m, 2m, -(4n+3)m, -2m}\,]_{n=0}^{\infty} \\ &= [0; m, \overline{2m-1, 1, (2n+1)m-1}\,]_{n=1}^{\infty}. \end{aligned} \tag{3.5}$$

If we set $m = \sqrt{uv}$ and multiply the left side of (3.5) and the first continued fraction on the right side of (3.5) by $\sqrt{v/u}$, we get

$$\begin{aligned} v(1-e^{-1/uv}) &= [0; \overline{(4n+1)u, 2v, -(4n+3)u, -2v}\,]_{n=0}^{\infty} \\ &= [0; u, \overline{2v-1, 1, (2n+1)u-1}\,]_{n=1}^{\infty}. \end{aligned} \tag{3.6}$$

We have not seen the continued fraction expansions at (3.5) and (3.6) elsewhere.

We are now ready to derive several new families of Hurwitzian continued fractions, using Corollary 6.

Theorem 10. *Let a, b, p, u and v be integers restricted in the case of each continued fraction below so that the partial quotients are all positive. Then*

$$\begin{aligned} &[0; \overline{p-1, 1, u(a+2nb)-1, p-1, 1, v(a+(2n+1)b)-1}\,]_{n=0}^{\infty} \\ &\qquad = \frac{1}{p} + \frac{1}{p}\sqrt{\frac{v}{u}}\,\frac{I_{(a/b)}(2/bp\sqrt{uv})}{I_{(a/b)-1}(2/bp\sqrt{uv})}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &[0; p-1, \overline{1, (4n+1)u-1, p, (4n+3)v-1, 1, p-2}\,]_{n=0}^{\infty} \\ &\qquad = \frac{1}{p} + \frac{1}{p}\sqrt{\frac{v}{u}}\tan\frac{1}{p\sqrt{uv}}, \end{aligned} \tag{3.8}$$

$$[0; \overline{p-1, 1, (4n+1)u-1, p-1, 1, (4n+3)v-1}\,]_{n=0}^{\infty} = \frac{1}{p} + \frac{1}{p}\sqrt{\frac{v}{u}} \tanh \frac{1}{p\sqrt{uv}}, \tag{3.9}$$

$$[0; p-1, \overline{1, (4n+1)u-1, p-1, 1, 2v-1, p, (4n+3)u-1,} \\ \overline{1, p-1, 2v-1, 1, p-2}\,]_{n=0}^{\infty} = \frac{1}{p} + v(1-e^{-1/up^2v}). \tag{3.10}$$

Proof. The claimed identities follow by applying the result in Corollary 6 to, in turn, (3.1), (3.2), (3.3) and (3.6) (replace u by up^2 in each case), and then removing the negative signs from the resulting continued fractions. □

Remark: Variants of each of these continued fraction identities could be produced by replacing some of the parameters in each expansion in Theorem 10 by their negatives, as in Corollary 2, but we do not consider that here.

3.1. **Finite continued fractions containing arithmetic progressions.** Here we find expressions for finite continued fractions of the form $[0; a, a+b, a+2b, a+3b, \cdots, a+(n-1)b]$ and $[0; a, c, a+b, c+d, a+2b, c+2d, \cdots, a+(n-1)b, c+(n-1)d]$, where a, b, c and d satisfy a simple algebraic relation. We first prove the following theorem.

Theorem 11. *Let*

$$\frac{P_n}{Q_n} := \frac{-c}{a} - \frac{c}{a+b} - \frac{c}{a+2b} - \cdots - \frac{c}{a+(n-1)b} \tag{3.11}$$

denote the n-th approximant of the continued fraction $K_{j=0}^{\infty} - c/(a+jb)$. Then

$$P_n = \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \binom{n-i}{i-1} (-c)^i \prod_{j=i}^{n-i} (a+j\,b), \tag{3.12}$$

$$Q_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} (-c)^i \prod_{j=i}^{n-1-i} (a+j\,b).$$

Proof. The statements are easily checked to be true for $n=1$ and $n=2$ (as usual, the empty product is taken to be equal to 1). Now suppose the statements are true for $n = 1, 2, \ldots k$.

$$\begin{aligned} P_{k+1} &= (a+kb)P_k - cP_{k-1} \\ &= (a+kb) \sum_{i=1}^{\lfloor (k+1)/2 \rfloor} \binom{k-i}{i-1} (-c)^i \prod_{j=i}^{k-i} (a+j\,b) \\ &\qquad - c \sum_{i=1}^{\lfloor k/2 \rfloor} \binom{k-1-i}{i-1} (-c)^i \prod_{j=i}^{k-1-i} (a+j\,b) \end{aligned}$$

$$= -c\prod_{j=1}^{k}(a+j\,b) + (a+kb)\sum_{i=2}^{\lfloor (k+1)/2\rfloor}\binom{k-i}{i-1}(-c)^i\prod_{j=i}^{k-i}(a+j\,b)$$

$$+\sum_{i=2}^{\lfloor k/2\rfloor+1}\binom{k-i}{i-2}(-c)^i\prod_{j=i-1}^{k-i}(a+j\,b). \tag{3.13}$$

If k is odd, then $\lfloor (k+1)/2\rfloor = \lfloor k/2\rfloor + 1 = \lfloor (k+2)/2\rfloor$ and

$$(a+kb)\binom{k-i}{i-1}(-c)^i\prod_{j=i}^{k-i}(a+j\,b) + \binom{k-i}{i-2}(-c)^i\prod_{j=i-1}^{k-i}(a+j\,b)$$

$$= (-c)^i\prod_{j=i}^{k-i}(a+j\,b)\frac{(k-i)!}{(i-2)!(k-2i+1)!}\left(\frac{a+kb}{i-1}+\frac{a+(i-1)b}{k-2i+2}\right)$$

$$= (-c)^i\prod_{j=i}^{k-i}(a+j\,b)\frac{(k-i)!}{(i-2)!(k-2i+1)!}\frac{(k-i+1)(a+(k-i+1)b)}{(i-1)(k-2i+2)}$$

$$= \binom{k+1-i}{i-1}(-c)^i\prod_{j=i}^{k+1-i}(a+j\,b),$$

$$\Longrightarrow P_{k+1} = \sum_{i=1}^{\lfloor (k+2)/2\rfloor}\binom{k+1-i}{i-1}(-c)^i\prod_{j=i}^{k+1-i}(a+j\,b).$$

If k is even, the extra $\lfloor k/2\rfloor+1$-th term at (3.13) provides the $\lfloor (k+2)/2\rfloor$-th term in the sum above. The proof of (3.12) for P_n now follows.

The proof for Q_n is virtually identical, and so is omitted. □

Corollary 7. *Let a and b be positive integers. Then*

$$\frac{1}{a}+\frac{1}{a+b}+\cdots+\frac{1}{a+(n-1)b} = \frac{\sum_{i=1}^{\lfloor (n+1)/2\rfloor}\binom{n-i}{i-1}\prod_{j=i}^{n-i}(a+j\,b)}{\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n-i}{i}\prod_{j=i}^{n-1-i}(a+j\,b)}. \tag{3.14}$$

Let f, g, h and k be integers such that $2gh = k(2f+h)$. Then

$$\frac{1}{f}+\frac{1}{g}+\frac{1}{f+h}+\frac{1}{g+k}+\cdots+\frac{1}{f+(n-1)h}+\frac{1}{g+(n-1)k} \tag{3.15}$$

$$= \frac{\sum_{i=1}^{n}\binom{2n-i}{i-1}\left(\frac{2g}{2\,f+h}\right)^{2n-i}\prod_{j=i}^{2n-i}\left(f+j\,\frac{h}{2}\right)}{\sum_{i=0}^{n}\binom{2n-i}{i}\left(\frac{2g}{2\,f+h}\right)^{2n-1-i}\prod_{j=i}^{2n-1-i}\left(f+j\,\frac{h}{2}\right)}.$$

Proof. The identity at (3.14) follows immediately, upon setting $c = -1$ in Theorem 11. For (3.15), it is easy to see that

$$\frac{-c}{a} - \frac{c}{a+b} - \frac{c}{a+2b} - \cdots - \frac{c}{a+(2n-1)b}$$
$$= \frac{1}{-a/c} + \frac{1}{a+b} + \frac{1}{-a/c-2b/c} + \frac{1}{a+3b} +$$
$$\cdots + \frac{1}{-a/c-(2n-2)b/c} + \frac{1}{a+(2n-1)b}.$$

Now make the substitutions

$$a = \frac{2\,f\,g}{2\,f+h}, \qquad b = \frac{g\,h}{2\,f+h}, \qquad c = -\frac{2g}{2f+h},$$

and the continued fraction at (3.15) is produced. The result follows, after some simple manipulations, upon making the same substitutions into the ratio P_{2n}/Q_{2n}, where P_{2n} and Q_{2n} are as defined at (3.12). □

Lehmer's result (1.2) easily follows from (3.14), upon re-indexing the numerator on the right side by replacing i with $i+1$, dividing top and bottom on the right side by $\prod_{j=0}^{n-1}(a+j\,b)$, performing some simple algebraic manipulations, and then letting $n \to \infty$.

Corollary 8. (*Lehmer* [12]) *Let a and b be positive integers. Then*

$$[0; a, a+b, a+2b, a+3b, \cdots] = \frac{1}{b}\frac{\displaystyle\sum_{k=0}^{\infty}\frac{b^{-2k}}{(a/b)_{k+1}k!}}{\displaystyle\sum_{k=0}^{\infty}\frac{b^{-2k}}{(a/b)_{k}k!}}.$$

4. Hurwitzian- and Tasoevian continued fractions with arbitrarily long quasi-period

We conclude by noting that the construction described in Corollary 6 can be iterated to produce both Hurwitzian- and Tasoevian continued fractions with arbitrary long quasi-period, with arbitrarily many free parameters and whose limits can be determined. We give one example, with seven free parameters and quasi-period of length 24, to illustrate this.

Theorem 12. *Let e, f, $p > 1$, $q > 1$, $r > 2$, $u > 1$ and $v > 1$ be positive integers. Let $E = ep^2q^4r^8$. Then*

$$[0; \overline{r-1, 1, q-1, r, p-1, 1, r-1, q-1, 1, r-1, eu^n-1, 1,}$$
$$\overline{r-2, 1, q-1, r-1, 1, p-1, r, q-1, 1, r-2, 1, fv^n-1}\,]_{n=1}^{\infty} \tag{4.1}$$
$$= \frac{1}{pq^2r^4} + \frac{1}{qr^2} + \frac{1}{r}$$

$$+\left(\frac{1}{Eu}-\frac{1}{E^2fu^2v+E}\right)\frac{\sum_{n=0}^{\infty}\dfrac{(Ef)^{-n}(uv)^{-n(n+3)/2}}{(1/uv;1/uv)_n(-1/Efu^3v^2;1/uv)_n}}{\sum_{n=0}^{\infty}\dfrac{(Ef)^{-n}(uv)^{-n(n+1)/2}}{(1/uv;1/uv)_n(-1/Efu^2v;1/uv)_n}}.$$

Proof. For ease of notation, let

$$f(e)=\left(\frac{1}{eu}-\frac{1}{e^2fu^2v+e}\right)\frac{\sum_{n=0}^{\infty}\dfrac{(ef)^{-n}(uv)^{-n(n+3)/2}}{(1/uv;1/uv)_n(-1/efu^3v^2;1/uv)_n}}{\sum_{n=0}^{\infty}\dfrac{(ef)^{-n}(uv)^{-n(n+1)/2}}{(1/uv;1/uv)_n(-1/efu^2v;1/uv)_n}},$$

so that, by Theorem 4,

$$[0;\overline{eu^n,fv^n}]_{n=1}^{\infty}=f(e).$$

Replace e with ep^2 and, by Corollary 6,

$$\left[0;\overline{p,-eu^n,-p,fv^n}\ \right]_{n=1}^{\infty}=\frac{1}{p}+f(ep^2).$$

Replace p with pq^2 and, again by Corollary 6,

$$\left[0;\overline{q,-p,-q,-eu^n,q,p,-q,fv^n}\ \right]_{n=1}^{\infty}=\frac{1}{q}+\frac{1}{pq^2}+f(ep^2q^4).$$

Repeat this step once more, by replacing q with qr^2, and then

$$\left[0;\overline{r,-q,-r,-p,r,q,-r,-eu^n,r,-q,-r,p,r,q,-r,fv^n}\ \right]_{n=1}^{\infty}$$
$$=\frac{1}{r}+\frac{1}{qr^2}+\frac{1}{pq^2r^4}+f(ep^2q^4r^8).$$

Finally, remove the negatives from the continued fraction and (4.1) follows. □

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Mathematics Department, Anderson Hall, West Chester University, West Chester, PA 19383

E-mail address: jmclaughl@wcupa.edu