A Generalization of Schröter's Formula

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To George Andrews, on his 80th Birthday

Abstract. We prove a generalization of Schröter's Formula to a product of an arbitrary number of Jacobi triple products. It is then shown that many of the well-known identities involving Jacobi triple products (for example the Quintuple Product Identity, the Septuple Product Identity, and Winquist's Identity) all then follow as special cases of this general identity. Various other general identities, for example certain expansions of $(q;q)_\infty$ and $(q;q)_\infty^k$, $k \geq 3$, as combinations of Jacobi triple products, are also proved.

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1. Foreword

I was happy to receive the email sent by the organizers of the Combinatory Analysis 2018 conference, reminding attendees that there would be a "Special Issue of the Annals of Combinatorics to honor George Andrews at the occasion of passing the milestone age of 80", and soliciting papers with "new or unpublished work relating to the mathematical interests of George Andrews".

I had previously done some work on extending Schröter's identity for a product of two Jacobi triple products to a product of arbitrarily many such products. When this email sent to conference attendees arrived from the organizers, it spurred me to complete the proof of the identity that I had found. The topic, Jacobi triple products, is certainly one that is frequently

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found in the papers of Professor Andrews, so I was happy to submit this paper to the conference proceedings.

2. Introduction

The Jacobi triple product identity, first proved by Jacobi [9], is one of the fundamental identities in q-series. It may be written (see, for example, [6, Equation (II.28), page 357]) as

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = \left(-qz, \frac{-q}{z}, q^2; q^2\right)_\infty . \tag{2.1}$$

For space saving reasons we will occasionally use the notation

$$\langle a; q^{2j} \rangle_\infty$$

to denote the triple product $(a, q^{2j}/a, q^{2j}; q^{2j})_\infty$.

This paper is concerned with identities in which products of Jacobi triple products are expanded into sums and products of other triple products. A well-known example of such an identity is the quintuple product identity (see Cooper's excellent paper [4] for the history of this identity and a survey of its various proofs). This identity may be written as

$$\begin{aligned}(z, q^2/z, q^2; q^2)_\infty (q^2z^2, q^2/z^2; q^4)_\infty &= \\ (-q^2z^3, -q^4/z^3, q^6; q^6)_\infty - z(-q^4z^3, -q^2/z^3, q^6; q^6)_\infty&. \end{aligned} \tag{2.2}$$

Remark: Strictly speaking, the factor $(q^2z^2, q^2/z^2; q^4)_\infty$ is not a triple product, but becomes so if we multiply on the left side by $(q^4; q^4)_\infty$, while the right side remains a sum/product combination of triple products if we multiply on the right side by the equivalent $(q^4, q^8, q^{12}; q^{12})_\infty$. A similar situation will hold for other identities in the paper.

A second example is given by the Septuple Product Identity, first found by Hirschhorn [7] (in fact, Hirschhorn found a two-parameter extension from which the Septuple Product Identity follows upon setting $a = -z/q$ and $b = -z^2/q$ in Equation (2.1) on page 32), and re-discovered by Farkas and Kra [5].

$$\begin{aligned}&\left(z, \frac{q^2}{z}, q^2; q^2\right)_\infty \left(-z, \frac{-q^2}{z}, q^2; q^2\right)_\infty \left(z, \frac{q^4}{z}, q^4; q^4\right)_\infty \\ &= (q^2, q^2, q^4; q^4)_\infty (q^8, q^{12}, q^{20}; q^{20})_\infty \\ &\qquad \times \left\{ \left(q^4z^5, \frac{q^{16}}{z^5}, q^{20}; q^{20}\right)_\infty + z^3 \left(q^{16}z^5, \frac{q^4}{z^5}, q^{20}; q^{20}\right)_\infty \right\} \\ &- (q^2, q^2, q^4; q^4)_\infty (q^4, q^{16}, q^{20}; q^{20})_\infty \\ &\qquad \times \left\{ z\left(q^8z^5, \frac{q^{12}}{z^5}, q^{20}; q^{20}\right)_\infty + z^2 \left(q^{12}z^5, \frac{q^8}{z^5}, q^{20}; q^{20}\right)_\infty \right\}.\end{aligned}$$

A third example is contained in Winquist's Identity (Winquist [10]):

$$\left(a, \frac{q^2}{a}, b, \frac{q^2}{b}, ab, \frac{q^2}{ab}, \frac{a}{b}, \frac{q^2 b}{a}, q^2, q^2, q^2, q^2; q^2\right)_\infty = (q^2, q^2; q^2)_\infty \Bigg[\left(a^3, \frac{q^6}{a^3}, q^6; q^6\right)_\infty \left\{ \left(b^3 q^2, \frac{q^4}{b^3}, q^6; q^6\right)_\infty - b\left(b^3 q^4, \frac{q^2}{b^3}, q^6; q^6\right)_\infty \right\} - \frac{a}{b}\left(b^3, \frac{q^6}{b^3}, q^6; q^6\right)_\infty \left\{ \left(a^3 q^2, \frac{q^4}{a^3}, q^6; q^6\right)_\infty - a\left(a^3 q^4, \frac{q^2}{a^3}, q^6; q^6\right)_\infty \right\} \Bigg].$$

In the present paper we prove an expansion for a product of k ($k \geq 3$) Jacobi triple products in terms of sums of products of other Jacobi triple products (Theorem 2.1 below), and then show that all of the identities above, and also various other identities, follow as special cases. The main theorem of the paper is the following.

Theorem 2.1. *Let $k \geq 1$ be a positive integer and let $n_1, n_2, \ldots, n_k$, N be positive integers such that $N = \operatorname{lcm}(n_1, n_2, ..., n_k)$, or a multiple thereof. For ease of notation, for $1 \leq i \leq k$ set $u_i := N/n_i$, $v_i := u_1 + u_2 + \cdots + u_i$, and $w_i := v_i + 1$. Let z, a, a_1, a_2 … a_k be non-zero complex numbers and suppose $|q| < 1$. Then*

$$\begin{aligned} &\left\langle -q^N az; q^{2N}\right\rangle_\infty \prod_{i=1}^{k} \left\langle -q^{n_i} a_i z; q^{2n_i}\right\rangle_\infty \\ &= \sum_{j_1=0}^{v_1} \sum_{j_2=0}^{v_2} \cdots \sum_{j_k=0}^{v_k} z^{j_k} q^{n_1 j_1^2 + n_2 (j_2 - j_1)^2 + n_3 (j_3 - j_2)^2 + \cdots + n_k (j_k - j_{k-1})^2} \\ &\quad a_1^{j_1} a_2^{j_2 - j_1} a_3^{j_3 - j_2} \ldots a_k^{j_k - j_{k-1}} \left\langle -q^{n_1 + N + 2n_1 j_1} \frac{a_1}{a}; q^{2(n_1 + N)}\right\rangle_\infty \\ &\prod_{i=2}^{k} \left\langle - q^{n_i (w_i w_{i-1} + 2 w_i j_{i-1} - 2 w_{i-1} j_i)} \frac{a\, a_1^{u_1} a_2^{u_2} \ldots a_{i-1}^{u_{i-1}}}{a_i^{w_{i-1}}}; q^{2 n_i w_i w_{i-1}} \right\rangle_\infty \\ &\qquad \times \left\langle -q^{N w_k + 2N j_k} a_1^{u_1} a_2^{u_2} \ldots a_k^{u_k} a z^{w_k}; q^{2N w_k}\right\rangle_\infty . \end{aligned} \tag{2.3}$$

Observe that the expansion (2.3) provides a w_k-dissection of the left side into powers of z that lie in arithmetic progressions modulo w_k.

The quintuple product identity, the septuple product identity and Winquist's identity, and others, all follow from special cases of the above identity. Other applications include expansions of Ramanujan theta functions, or powers of these, as sums and products of Jacobi triple products. As an example of one of these latter identities, we have that, for an arbitrary integer $k \geq 3$,

$$\begin{aligned} &(q;q)_\infty^k = \\ &\sum_{j_1=0}^{1} \cdots \sum_{j_{k-1}=0}^{k-1} (-1)^{j_{k-1}} q^{3(j_1^2 + j_2^2 + \cdots + j_{k-2}^2) + j_{k-1}(3 j_{k-1} + 1)/2 - 3(j_1 j_2 + \cdots + j_{k-2} j_{k-1})} \\ &\qquad \times \left\langle -q^{3 + 3 j_1}; q^6\right\rangle_\infty \left\langle (-1)^{k+1} q^{2k + 3 j_{k-1}}; q^{3k}\right\rangle_\infty \end{aligned}$$

$$\times \prod_{i=2}^{k-1} \left\langle -q^{3i(i+1)/2+3(i+1)j_{i-1}-3ij_i}; q^{3i(i+1)} \right\rangle_\infty .$$

Remark: Cao [3] proves a quite general theorem (Theorem 1.4) which also exhibits a product of arbitrarily many Jacobi triple products as a sum containing other Jacobi triple products. Cao's theorem is more general in the sense that it allows for a greater variety of expansions, but in full generality appears more restrictive in the sense that for such identity, it must also be shown that the entries of a certain associated matrix satisfy certain conditions. We had initially though that the result in the present paper was independent of Cao's result. However, it was pointed out by one of the referees that the key induction step in the proof of our Theorem 2.1, namely, Corollary 3.8, is actually a special case of Corollary 2.2 in Cao's paper [3], and thus that our result in Theorem 2.1 could have been derived from the results in Cao's paper [3], by following the appropriate path and making the appropriate specializations.

3. Extensions of Schröter's Identity.

Before coming to the main theorem and its consequences, we first consider Schröter's Identity, and also state some elementary extensions. The methods of proof will also preview the methods used to prove the main theorem in the next section.

Schröter's identity (see [2, page 111]),

$$\begin{aligned} &\left\langle -q^{n_1}a; q^{2n_1} \right\rangle_\infty \left\langle -q^{n_2}b; q^{2n_2} \right\rangle_\infty \\ &\qquad = \sum_{j=0}^{n_1+n_2-1} q^{n_1 j^2} a^j \left\langle \frac{-q^{n_1+n_2+2n_1 j}a}{b}; q^{2(n_1+n_2)} \right\rangle_\infty \\ &\qquad\qquad \times \left\langle -q^{(n_1+n_2+2j)n_1 n_2} a^{n_2} b^{n_1}; q^{2(n_1+n_2)n_1 n_2} \right\rangle_\infty , \end{aligned} \tag{3.1}$$

first appeared in Schröter's 1854 dissertation.

In Lemma 3.3 we introduce a variable z as a "book-keeping" device by replacing a with az and b with bz, and also give a proof of a slight extension of Schröter's identity by introducing an integer variable m into the summation range (Schröter's original identity is the case $m = 1$ of the identity in Lemma 3.3), and then use this result in conjunction with Lemma 3.1 to derive a more general extension.

We begin by recalling the following well-known elementary identity.

Lemma 3.1. *Let p be a positive integer and let q and z be complex numbers with $z \neq 0$ and $|q| < 1$. Then*

$$\left\langle -qz; q^2 \right\rangle_\infty = \sum_{j=0}^{p-1} q^{j^2} z^j \left\langle -q^{p^2+2pj} z^p; q^{2p^2} \right\rangle_\infty . \tag{3.2}$$

Proof. The proof follows directly from (2.1), upon breaking the sum on the left side into p sums, in each of which the exponents n all lie in the same arithmetic progression modulo p, and then applying (2.1) to each sum. □

Before coming to the extension of Schröter's identity, we also need a preliminary lemma.

Lemma 3.2. *If c is a non-zero complex number, n is any positive integer, j is any integer, and $|q| < 1$, then*

$$(cq^{n+2jn}, q^{n-2jn}/c; q^{2n})_\infty = (cq^n; q^n/c; q^{2n})_\infty \left(\frac{-1}{c}\right)^j \frac{1}{q^{nj^2}}. \tag{3.3}$$

Proof. The statement is clearly true if $j = 0$. If $j > 0$, then

$$(cq^{n+2jn}, q^{n-2jn}/c; q^{2n})_\infty = (cq^n; q^n/c; q^{2n})_\infty \frac{(q^{n-2jn}/c; q^{2n})_j}{(cq^n; q^{2n})_j},$$

and the result follows for $j > 0$ upon applying the identity (see [6, Identity (I.8), page 351])

$$(aq^{-j}; q)_j = (q/a; q)_j \left(\frac{-a}{q}\right)^j q^{-j(j-1)/2}$$

to $(q^{n-2jn}/c; q^{2n})_j$ (with $a = q^n/c$ and q replaced with q^{2n}). The result for $j < 0$ follows from the $j > 0$ case, after replacing j with $-j$ and c with $1/c$. □

Lemma 3.3. *(An extension of Schröter's Identity) Let a, b and z be non-zero complex numbers, let q be a complex number with $|q| < 1$, and let $m \geq 1$ be an integer. Then*

$$\begin{aligned}
&\left\langle -q^{n_1}az; q^{2n_1}\right\rangle_\infty \left\langle -q^{n_2}bz; q^{2n_2}\right\rangle_\infty \\
&\quad = \sum_{j=0}^{m(n_1+n_2)-1} q^{n_1 j^2} (az)^j \left\langle \frac{-q^{n_1+n_2+2n_1 j} a}{b}; q^{2(n_1+n_2)}\right\rangle_\infty \\
&\times \left\langle -q^{(m(n_1+n_2)+2j)n_1 n_2 m} a^{mn_2} b^{mn_1} z^{m(n_1+n_2)}; q^{2(n_1+n_2)n_1 n_2 m^2}\right\rangle_\infty .
\end{aligned} \tag{3.4}$$

Proof. After two applications of the Jacobi triple product identity,

$$\begin{aligned}
F_1(z) &:= \left\langle -q^{n_1}az; q^{2n_1}\right\rangle_\infty \left\langle -q^{n_2}bz; q^{2n_2}\right\rangle_\infty \\
&= \sum_{m_1, m_2 \in \mathbb{Z}} q^{n_1 m_1^2} (az)^{m_1} \frac{q^{n_2 m_2^2}}{(bz)^{m_2}} \\
&= \sum_{t, m_2 \in \mathbb{Z}} (az)^t q^{n_1(t^2+2tm_2+m_2^2)} \frac{q^{n_2 m_2^2} a^{m_2}}{b^{m_2}} \qquad (t = m_1 - m_2) \\
&= \sum_{t, m_2 \in \mathbb{Z}} (az)^t q^{n_1 t^2} q^{(n_1+n_2)m_2^2} \left(\frac{aq^{2n_1 t}}{b}\right)^{m_2}
\end{aligned}$$

$$= \sum_{t\in\mathbb{Z}} (az)^t q^{n_1 t^2} \left\langle \frac{-q^{n_1+n_2+2n_1 t} a}{b}; q^{2(n_1+n_2)} \right\rangle_\infty .$$

Now set $t = m(n_1+n_2)r + j$, $r \in \mathbb{Z}$, $0 \le j \le m(n_1+n_2)-1$, and apply Lemma 3.2 (with n_1+n_2, $n_1 mr$ and $-ab^{-1}q^{2n_1 j}$ instead of n, j and c, respectively) to the triple products to get

$$F_1(z) = \sum_{j=0}^{m(n_1+n_2)-1} q^{n_1 j^2} (az)^j \left\langle \frac{-q^{n_1+n_2+2n_1 j} a}{b}; q^{2(n_1+n_2)} \right\rangle_\infty \times \sum_{r\in\mathbb{Z}} q^{m^2(n_1+n_2)n_1 n_2 r^2} \left(a^{n_2} b^{n_1} z^{n_1+n_2} q^{2n_1 n_2 j}\right)^{mr}.$$

The result follows after one further application of the Jacobi triple product identity. □

Theorem 3.4. *(A second extension of Schröter's Identity) Let a, b and z be non-zero complex numbers, let q be a complex number with $|q| < 1$, and let $m \ge 1$, $n_1 \ge 1$, $n_2 \ge 1$ and $p \ge 1$ be integers. Then*

$$\begin{aligned} &\left\langle -q^{n_1} az; q^{2n_1} \right\rangle_\infty \left\langle -q^{n_2} bz; q^{2n_2} \right\rangle_\infty \\ &= \sum_{j_1=0}^{m(n_1+p^2 n_2)-1} \sum_{j_2=0}^{p-1} q^{n_1 j_1^2 + j_2^2 n_2} a^{j_1} b^{j_2} z^{j_1+j_2} \\ &\left\langle \frac{-q^{n_1+p^2 n_2 + 2n_1 j_1 - 2p j_2 n_2} a}{b^p z^{p-1}}; q^{2(n_1+p^2 n_2)} \right\rangle_\infty \\ &\Big\langle -a^{mp^2 n_2} b^{mn_1 p} z^{mp(n_1+pn_2)} q^{mn_1 p n_2 \left(2j_1 p + 2j_2 + mp\left(n_1+p^2 n_2\right)\right)}; \\ &\qquad q^{2m^2 n_1 p^2 n_2 \left(n_1+p^2 n_2\right)} \Big\rangle_\infty . \end{aligned} \tag{3.5}$$

Proof. Apply Lemma 3.1 to the product $\left(-q^{n_2} bz, -q^{n_2}/bz, q^{2n_2}; q^{2n_2}\right)_\infty$, and then apply Lemma 3.3 to each pair of triple products in the resulting expression. □

Remark: For the statement of Lemma 3.3, and Theorem 3.4, the presence of the z variable is not actually necessary, as it could be absorbed into the a and b variables, without affecting the generality of the result. However, its usefulness derives from the fact that the right side of (3.4) provides a $m(n_1+n_2)$-dissection of the left side into $m(n_1+n_2)$ functions in each of which the powers of z all lie in the same arithmetic progression modulo $m(n_1+n_2)$. For this reason, we retain the variable z in Theorem 3.4, and elsewhere throughout the paper (see, for example, the proof of Corollary 4.6, where this dissection proves useful).

We note that in the case where $n_1 | n_2$, there exists a second family of expansions that do not come directly from Theorem 3.4.

Corollary 3.5. *Let a, b z, q, m, p, n_1 and n_2 be as in Theorem 3.4, with the additional requirement that $n_1|n_2$. Then*

$$\begin{aligned}&\left\langle -q^{n_1}az;q^{2n_1}\right\rangle_\infty \left\langle -q^{n_2}bz;q^{2n_2}\right\rangle_\infty\\ &\quad=\sum_{j_1=0}^{m(1+p^2n_2/n_1)-1}\sum_{j_2=0}^{p-1} q^{n_1j_1^2+n_2j_2^2}a^{j_1}b^{j_2}z^{j_1+j_2}\\ &\quad\left\langle \frac{-q^{n_1+p^2n_2+2n_1j_1-2pj_2n_2}a}{b^pz^{p-1}};q^{2(n_1+p^2n_2)}\right\rangle_\infty\\ &\quad\Big\langle -a^{mp^2n_2/n_1}b^{mp}z^{mp(1+pn_2/n_1)}q^{mpn_2\left(2j_1p+2j_2+mp\left(1+p^2n_2/n_1\right)\right)};\\ &\qquad q^{2m^2p^2n_2\left(1+p^2n_2/n_1\right)}\Big\rangle_\infty. \end{aligned}\tag{3.6}$$

Proof. Write

$$\left\langle -q^{n_2}bz;q^{2n_2}\right\rangle_\infty=\left\langle -(q^{n_1})^{n_2/n_1}bz;(q^{n_1})^{2n_2/n_1}\right\rangle_\infty,$$

and then apply Theorem 3.4, with n_1 replaced with 1, n_2 replaced with n_2/n_1 and q replaced with q^{n_1}. □

Theorem 3.4 is more general in the sense that it holds also when $n_1 \nmid n_2$. However, when $n_1|n_2$, Corollary 3.5 is actually the stronger result as it implies Theorem 3.4 in this case (replace m with mn_1 in Corollary 3.5).

Remarks: (1) Cao ([3, Theorem 2.3, Equation (2.50)]), using a different approach, has given a Generalized Schröter's Formula for a product of two Jacobi triple products, a formula which implies our identity (3.4), but not (3.5) (or at least not without additional transformations).

(2) Even though no applications of (3.4), (3.5) and (3.6) are given in the present paper with $m > 1$ or $p > 1$, they are included for the sake of completeness.

We note that the quintuple product identity also follows from Schröter's Theorem. The quintuple product identity is usually written in the form

$$(-z,-q/z,q;q)_\infty(qz^2,q/z^2;q^2)_\infty=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}z^{3n}(1+zq^n).$$

Upon replacing q with q^2, z with $-z$, using the Jacobi triple product identity to sum the resulting series on the right side, this identity may be restated in the form given at (2.2), and we show that it follows from Corollary 3.5.

Corollary 3.6. *Let z be a non-zero complex number, and suppose $|q| < 1$. Then* (2.2) *holds.*

Proof. In (3.4), set $m = 1$, $n_2 = 2$, $a = -1/q^3$, $b = -1/z^3$, and (2.2) follows after some simple manipulations, after using the fact that $(q^4, q^8, q^{12}; q^{12})_\infty = (q^4; q^4)_\infty$. □

Schröter's theorem and its various extensions for a product of two Jacobi triple products naturally leads to the following question. Given a product of k ($k \geq 3$, k an integer) Jacobi triple products,

$$F(z) := \prod_{i=1}^{k} \left\langle -q^{n_i} a_i z; q^{2n_i} \right\rangle_\infty ,$$

and we M-dissect $F(z)$ by writing

$$F(z) = \sum_{j=0}^{M-1} z^j F_j(z^M),$$

for some integer M, can an explicit representation of each $F_j(z^M)$ be given? We gave an affirmative answer to this question in Theorem 2.1, and prove this theorem in the next section, in the case where each $n_i|N$ (with $M = N/n_1 + \cdots + N/n_k + 1$).

To this end, an identity which follows from a special case of the next identity, due to Cao [3], is needed. We state this result of Cao in terms of q-products, rather than using Ramanujan's theta function $f(a,b)$, as Cao did.

Proposition 3.7. (*Cao,* [3, Corollary 2.2]) *If* $|ab| < 1$ *and* $(cd) = (ab)^{k_1 k_2}$, *where both* k_1 *and* k_2 *are positive integers, then*

$$(-a,-b,ab;ab)_\infty(-c,-d,cd;cd)_\infty = \sum_{r=0}^{k_1+k_2-1} (ab)^{r^2/2} \left(\frac{a}{b}\right)^{r/2}$$
$$\times \left((ab)^{k_1^2/2+k_1 r} \left(\frac{a}{b}\right)^{k_1/2} c, (ab)^{k_1^2/2-k_1 r} \left(\frac{b}{a}\right)^{k_1/2} d, (ab)^{k_1^2} cd; (ab)^{k_1^2} cd \right)_\infty$$
$$\times \left((ab)^{k_2^2/2+k_2 r} \left(\frac{a}{b}\right)^{k_2/2} d, (ab)^{k_2^2/2-k_2 r} \left(\frac{b}{a}\right)^{k_2/2} c, (ab)^{k_2^2} cd; (ab)^{k_2^2} cd \right)_\infty . \tag{3.7}$$

The special case that is needed may be stated as follows.

Corollary 3.8. *Let* j' *be an integer and let* n, N *and* w' *be positive integers such that* $n|N$. *Let* a, e, z *and* q *be non-zero complex numbers with* $|q| < 1$. *Define*

$$u := \frac{N}{n}, \qquad w := u + w', \qquad v := w - 1.$$

Then

$$\left\langle -ezq^n; q^{2n} \right\rangle_\infty \left\langle -az^{w'} q^{Nw'+2j'N}; q^{2Nw'} \right\rangle_\infty = \sum_{j=0}^{v} e^{j-j'} q^{n(j-j')^2} z^{j-j'}$$
$$\times \left\langle \frac{-a}{e^{w'}} q^{nw'w+2n(j'w-jw')}; q^{2nw'w} \right\rangle_\infty \left\langle -ae^u z^w q^{2jN+Nw}; q^{2Nw} \right\rangle_\infty . \tag{3.8}$$

Proof. In Proposition 3.7, set $a = ezq^n$, $b = q^n/(ez)$, $c = q^{N(w'-2j')}/(az^{w'})$, $d = az^{w'}q^{N(w'+2j')}$, $k_1 = u$ and $k_2 = w'$. Then it can be seen that $(cd) = (ab)^{k_1k_2}$, and that the left side of (3.7) becomes the left side of (3.8). The right side of (3.7) becomes

$$\sum_{r=0}^{v} q^{nr^2}(ez)^r \langle -ae^u z^w q^{N(w+2r+2j')}; q^{2Nw}\rangle_\infty \times \langle -\frac{e^{w'}}{a} q^{nww'+2n(w'r-j'w)+2nj'w'}; q^{2nww'}\rangle.$$

The result follows upon, in turn, replacing r with $r - j'$ (so that the interval of summation is also changed to one of another w consecutive integers), using the division algorithm (with r and w) to write each of the resulting new r values in the form $r = mw + j$ for some integers j and m with $0 \leq j \leq w - 1$, and finally applying (3.3) to each of the terms in the resulting sum. □

4. Main Result and its Implications

We now come to the proof of the main result of the paper. The proof is essentially a simple induction argument using identities (3.6) and (3.8).

Remark: It should be pointed out that attempting to iterate Schröter's original identity (the case $m = 1$ of the identity in Lemma 3.3) does not appear to easily lead to any result similar to that in Theorem 2.1.

Proof of Theorem 2.1. The proof is by induction on k. If $k = 1$, then (2.3) becomes

$$\left\langle -q^N az; q^{2N}\right\rangle_\infty \left\langle -q^{n_1}a_1 z; q^{2n_1}\right\rangle_\infty = \sum_{j_1=0}^{v_1} z^{j_1} q^{n_1 j_1^2} a_1^{j_1} \left\langle -q^{n_1+N+2n_1j_1}\frac{a_1}{a}; q^{2(n_1+N)}\right\rangle_\infty \times \left\langle -q^{Nw_1+2Nj_1} a_1^{u_1} az^{w_1}; q^{2Nw_1}\right\rangle_\infty .$$

However, this is simply identity (3.6), with $n_2 = N$, $b = a$, $m = p = 1$, upon recalling that $v_1 = u_1 = N/n_1$ and $w_1 = v_1 + 1$.

Now suppose (2.3) holds for $k = 1, 2, \ldots r$. Now consider the left side of (2.3) with $k = r + 1$, so that after employing the $k = r$ case on the first r Jacobi triple products,

$$\left\langle -q^N az; q^{2N}\right\rangle_\infty \prod_{i=1}^{r+1} \left\langle -q^{n_i}a_i z; q^{2n_i}\right\rangle_\infty = \sum_{j_1=0}^{v_1}\sum_{j_2=0}^{v_2}\cdots\sum_{j_r=0}^{v_r} z^{j_r} q^{n_1j_1^2+n_2(j_2-j_1)^2+n_3(j_3-j_2)^2+\cdots+n_r(j_r-j_{r-1})^2}$$
$$a_1^{j_1}a_2^{j_2-j_1}a_3^{j_3-j_2}\ldots a_r^{j_r-j_{r-1}} \left\langle -q^{n_1+N+2n_1j_1}\frac{a_1}{a}; q^{2(n_1+N)}\right\rangle_\infty$$

$$\prod_{i=2}^{r}\left\langle -q^{n_i(w_iw_{i-1}+2w_ij_{i-1}-2w_{i-1}j_i)}\frac{a\,a_1^{u_1}a_2^{u_2}\dots a_{i-1}^{u_{i-1}}}{a_i^{w_{i-1}}};q^{2n_iw_iw_{i-1}}\right\rangle_\infty$$
$$\times\left\langle -q^{Nw_r+2Nj_r}a_1^{u_1}a_2^{u_2}\dots a_r^{u_r}az^{w_r};q^{2Nw_r}\right\rangle_\infty\left\langle -q^{n_{r+1}}a_{r+1}z;q^{2n_{r+1}}\right\rangle_\infty. \tag{4.1}$$

Identity (3.8) is now applied to the final two triple products on the right side of (4.1) above. In this identity, n is replaced with n_{r+1}, j' with j_r, j with j_{r+1}, e with a_{r+1}, a with $aa_1^{u_1}a_2^{u_2}\dots a_r^{u_r}$ and w' with w_r. Hence, in the notation of Theorem 2.1, u takes the value $N/n_{r+1}=u_{r+1}$, w takes the value $u+w'=u_{r+1}+w_r=w_{r+1}$, and v takes the value $w-1=w_{r+1}-1=v_{r+1}$. After these substitutions are made, then (3.8) gives that

$$\left\langle -q^{n_{r+1}}a_{r+1}z;q^{2n_{r+1}}\right\rangle_\infty\left\langle -q^{Nw_r+2Nj_r}a_1^{u_1}a_2^{u_2}\dots a_r^{u_r}az^{w_r};q^{2Nw_r}\right\rangle_\infty$$
$$=\sum_{j_{r+1}=0}^{v_{r+1}}a_{r+1}^{j_{r+1}-j_r}q^{n_{r+1}(j_{r+1}-j_r)^2}z^{j_{r+1}-j_r}$$
$$\left\langle -q^{n_{r+1}(w_{r+1}w_r+2w_{r+1}j_r-2w_rj_{r+1})}\frac{a\,a_1^{u_1}a_2^{u_2}\dots a_r^{u_r}}{a_{r+1}^{w_r}};q^{2n_{r+1}w_{r+1}w_r}\right\rangle_\infty$$
$$\times\left\langle -q^{Nw_{r+1}+2Nj_{r+1}}a_1^{u_1}a_2^{u_2}\dots a_r^{u_r}a_{r+1}^{u_{r+1}}az^{w_{r+1}};q^{2Nw_{r+1}}\right\rangle_\infty. \tag{4.2}$$

The substitution of the right side of (4.2) into (4.1) to replace the left side of (4.2) gives that (2.3) holds for $k=r+1$, and thus by induction that it is true for all integers $k\geq 1$. This concludes the proof of Theorem 2.1. □

Corollary 4.1. *Let z, a_1, a_2 … a_k be non-zero complex numbers and suppose $|q|<1$. Then*

$$\prod_{i=1}^{k}\left\langle -qa_iz;q^2\right\rangle_\infty =$$
$$\sum_{j_1=0}^{1}\sum_{j_2=0}^{2}\cdots\sum_{j_{k-1}=0}^{k-1}z^{j_{k-1}}q^{j_1^2+(j_2-j_1)^2+\dots+(j_{k-1}-j_{k-2})^2}a_1^{j_1}a_2^{j_2-j_1}\dots a_{k-1}^{j_{k-1}-j_{k-2}}$$
$$\left\langle -q^{2+2j_1}\frac{a_1}{a_k};q^4\right\rangle_\infty\left\langle -q^{k+2j_{k-1}}a_1a_2\dots a_kz^k;q^{2k}\right\rangle_\infty$$
$$\times\prod_{i=2}^{k-1}\left\langle -q^{i(i+1)+2(i+1)j_{i-1}-2ij_i}\frac{a_ka_1a_2\dots a_{i-1}}{a_i^i};q^{2i(i+1)}\right\rangle_\infty. \tag{4.3}$$

Proof. Replace k with $k-1$ in Theorem 2.1, and then set $a=a_k$, $n_1=n_2=\dots=n_{k-1}=N=1$, so that each $u_i=1$, $v_i=i$ and $w_i=i+1$. □

Remarks: 1) Note that the sum (4.3) contains $k!$ terms, each with k Jacobi triple products.
2) Since the left side of (4.3) is invariant under any permutation of the numbers $a_1,\dots,a_k$, so is the right side.
3) The appearance of z in (4.3) is essentially a "book-keeping" device, as

without loss of generality each a_i could be replaced with a_i/z (or, equivalently, set $z = 1$).
4) Upon setting each $a_i = 1$, we get an expression for $(-qz, -q/z, q^2; q^2)_\infty^k$, $k \geq 3$.

$$\langle -qz; q^2\rangle_\infty^k = \sum_{j_1=0}^{1}\sum_{j_2=0}^{2}\cdots\sum_{j_{k-1}=0}^{k-1} z^{j_{k-1}} q^{j_1^2+(j_2-j_1)^2+\cdots+(j_{k-1}-j_{k-2})^2}\left\langle -q^{2+2j_1}; q^4\right\rangle_\infty \times \left\langle -q^{k+2j_{k-1}}z^k; q^{2k}\right\rangle_\infty \prod_{i=2}^{k-1}\left\langle -q^{i(i+1)+2(i+1)j_{i-1}-2ij_i}; q^{2i(i+1)}\right\rangle_\infty. \quad (4.4)$$

This identity provides expansions in terms of triple products for Ramanujan's theta functions, $f(-q) = \langle q; q^3\rangle_\infty = (q;q)_\infty$, $\phi(q) = \langle -q; q^2\rangle_\infty$ and $\psi(q) = \langle -q; q^4\rangle_\infty$. We give one example.

Corollary 4.2. *If $|q| < 1$, then*

$$f(-q)^k = (q;q)_\infty^k = \sum_{j_1=0}^{1}\cdots\sum_{j_{k-1}=0}^{k-1}(-1)^{j_{k-1}} q^{3(j_1^2+j_2^2+\cdots+j_{k-2}^2)+j_{k-1}(3j_{k-1}+1)/2-3(j_1j_2+j_2j_3+\cdots+j_{k-2}j_{k-1})} \times \left\langle -q^{3+3j_1}; q^6\right\rangle_\infty \left\langle (-1)^{k+1}q^{2k+3j_{k-1}}; q^{3k}\right\rangle_\infty \times \prod_{i=2}^{k-1}\left\langle -q^{3i(i+1)/2+3(i+1)j_{i-1}-3ij_i}; q^{3i(i+1)}\right\rangle_\infty. \quad (4.5)$$

Proof. In (4.4), replace q with $q^{3/2}$, and let $z = -q^{1/2}$. □

Remark: The squares in the exponent of q that precede the infinite products in (4.4) have been multiplied out and the terms rearranged, to make it more explicit that, after the replacement of q with $q^{3/2}$, that the new exponent is indeed integral.

As a second illustration of (2.3), we exhibit the $k = 3$ case of the identity explicitly, and show that it implies the quintuple product identity. This identity was also stated [3, Equation (3.2)] by Cao.

Corollary 4.3. *(Extended Quintuple Product Identity) If $a, b, c, z \neq 0$ and $|q| < 1$, then*

$$\left\langle -qaz; q^2\right\rangle_\infty \left\langle -qbz; q^2\right\rangle_\infty \left\langle -qcz; q^2\right\rangle_\infty = \left\langle -\frac{q^2a}{c}; q^4\right\rangle_\infty \left\{ \left\langle -\frac{q^6ac}{b^2}; q^{12}\right\rangle_\infty \left\langle -q^3abcz^3; q^6\right\rangle_\infty + qbz\left\langle -\frac{q^2ac}{b^2}; q^{12}\right\rangle_\infty \left\langle -q^5abcz^3; q^6\right\rangle_\infty + q^4b^2z^2\left\langle -\frac{ac}{q^2b^2}; q^{12}\right\rangle_\infty \left\langle -q^7abcz^3; q^6\right\rangle_\infty \right\} + \frac{q^2a}{b}\left\langle -\frac{q^4a}{c}; q^4\right\rangle_\infty$$

$$\left\{ \left\langle -\frac{q^{12}ac}{b^2}; q^{12} \right\rangle_\infty \left\langle -q^3abcz^3; q^6 \right\rangle_\infty + \frac{bz}{q} \left\langle -\frac{q^8ac}{b^2}; q^{12} \right\rangle_\infty \left\langle -q^5abcz^3; q^6 \right\rangle_\infty \right.$$
$$\left. + b^2z^2 \left\langle -\frac{q^4ac}{b^2}; q^{12} \right\rangle_\infty \left\langle -q^7abcz^3; q^6 \right\rangle_\infty \right\}. \quad (4.6)$$

Proof. This follows after some slight rearrangement of terms in (4.3), after substituting $k = 3$ and letting $a_1 = a$, $a_2 = b$ and $a_3 = c$. □

Recall from (2.2) that the quintuple product identity may be written as

$$(z, q^2/z, qz, q/z, -qz, -q/z, q^2, q^2, q^2; q^2)_\infty =$$
$$(q^2; q^2)^2_\infty \{(-q^2z^3, -q^4/z^3, q^6; q^6)_\infty - z(-q^4z^3, -q^2/z^3, q^6; q^6)_\infty\}.$$

However, this follows from (4.6) upon setting $a = -1$, $b = -1/q$ and $c = 1$, after some elementary infinite product manipulations.

The septuple product identity also follows from Theorem 2.1, but the proof is less direct, as it also needs Schröter's identity. We will prove the septuple product identity in the following form.

Corollary 4.4. *(Septuple Product Identity) Let z and q be complex numbers, with $z \neq 0$ and $|q| < 1$. Then*

$$\left\langle z; q^2 \right\rangle_\infty \left\langle -z; q^2 \right\rangle_\infty \left\langle z; q^4 \right\rangle_\infty$$
$$= \left\langle q^2; q^4 \right\rangle_\infty \left\langle q^8; q^{20} \right\rangle_\infty \left\{ \left\langle q^4z^5; q^{20} \right\rangle_\infty + z^3 \left\langle q^{16}z^5; q^{20} \right\rangle_\infty \right\}$$
$$- \left\langle q^2; q^4 \right\rangle_\infty \left\langle q^4; q^{20} \right\rangle_\infty \left\{ z \left\langle q^8z^5; q^{20} \right\rangle_\infty + z^2 \left\langle q^{12}z^5; q^{20} \right\rangle_\infty \right\}. \quad (4.7)$$

Proof. In (2.3), set $k = 2$, $n_1 = n_2 = 1$ and $N = 2$ (so that $u_1 = u_2 = 2$, $u_3 = 1$, $v_1 = 2, v_2 = 4$, $w_1 = 3$ and $w_2 = 5$), $z = 1$, $a_1 = -z/q$, $a_2 = z/q$ and $a_3 = -z/q^2$. Then the left side of (2.3) becomes the left side of (4.7), and the right side of (2.3) becomes

$$\sum_{j_2=0}^{4} q^{j_2^2 - j_2} z^{j_2} \langle q^{4+4j_2} z^5; q^{20} \rangle$$
$$\times \sum_{j_1=0}^{2} q^{2j_1^2 - 2j_1 j_2} (-1)^{j_1} \langle -q^{4+2j_1}; q^6 \rangle \langle q^{14-6j_2+10j_1}; q^{30} \rangle$$

It is easy to show that the inner sum is zero in the case $j_2 = 4$ and that proving (4.7) then comes down to proving the pair of identities

$$\langle q^2; q^4 \rangle \langle q^8; q^{20} \rangle = \langle -q^4; q^6 \rangle \langle q^{14}; q^{30} \rangle \quad (4.8)$$
$$- q^2 \langle -1; q^6 \rangle \langle q^{24}; q^{30} \rangle - q^2 \langle -q^2; q^6 \rangle \langle q^4; q^{30} \rangle,$$
$$\langle q^2; q^4 \rangle \langle q^4; q^{20} \rangle = \langle -1; q^6 \rangle \langle q^{18}; q^{30} \rangle \quad (4.9)$$
$$- \langle -q^2; q^6 \rangle \langle q^8; q^{30} \rangle - q^2 \langle -q^2; q^6 \rangle \langle q^{28}; q^{30} \rangle.$$

Let $g(m,n) := q^{2n^2+10m^2+2m}(-1)^{m+n}$. We use the Jacobi triple product identity to write the infinite product on the left side of identity (4.8) as an infinite series. We next use a method similar to that of Hirschhorn in [8] to first sum diagonally, and then divide the diagonal sums into six congruence classes. This gives

$$\begin{aligned}\langle q^2;q^4\rangle\langle q^8;q^{20}\rangle &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(m,n) = \sum_{k=-\infty}^{\infty}\sum_{m+n=k} g(m,n)\\ &= \sum_{j=0}^{5}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} g(s-r, 5s+r+j)\\ &= \sum_{j=0}^{5}(-1)^{6s+j}q^{2j^2}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} q^{12r^2+(4j-2)r+60s^2+(20j+2)s}.\end{aligned} \tag{4.10}$$

We similarly expand the right side of identity (4.8) to get

$$\begin{aligned}&\sum_{u,v=-\infty}^{\infty} q^{3u^2+u+15v^2+v}(-1)^v\\ &- q^2\sum_{u,v=-\infty}^{\infty} q^{3u^2+3u+15v^2+9v}(-1)^v - q^2\sum_{u,v=-\infty}^{\infty} q^{3u^2+u+15v^2+11v}(-1)^v.\end{aligned}$$

We next expand each the three sums into four sums by setting $u = 2r$ and $2r+1$ and $v = 2s$ and $2s+1$. By comparing the resulting twelve sums with the expression (4.10) (after possibly replacing r with $r \pm 1$ and/or s with $s \pm 1$ in some cases), it can be seen that proving identity (4.8) now depends on proving that

$$\begin{aligned}&q^6\langle -q^{10};q^{24}\rangle\langle -q^{22};q^{120}\rangle - q^4\langle -q^2;q^{24}\rangle\langle -q^{38};q^{120}\rangle\\ &\quad + q^8\langle -q^6;q^{24}\rangle\langle -q^{18};q^{120}\rangle - q^2\langle -q^6;q^{24}\rangle\langle -q^{42};q^{120}\rangle\\ &\quad - q^{14}\langle -q^{10};q^{24}\rangle\langle -q^2;q^{120}\rangle + q^2\langle -q^2;q^{24}\rangle\langle -q^{58};q^{120}\rangle = 0.\end{aligned}$$

However, this follows from Schröter's identity (3.1), by setting $n_1 = 1$, $n_2 = 5$, $a = -1/q$ and $b = -q^4$, and then replacing q with q^2.

The proof of the second identity (4.9) proceeds similarly, except at the end it depends on proving the identity

$$\begin{aligned}&-\langle -q^{10};q^{24}\rangle\langle -q^{46};q^{120}\rangle - q^{12}\langle -q^6;q^{24}\rangle\langle -q^6;q^{120}\rangle\\ &\quad + q^{10}\langle -q^2;q^{24}\rangle\langle -q^{24};q^{120}\rangle + q^4\langle -q^{10};q^{24}\rangle\langle -q^{26};q^{120}\rangle\\ &\quad + \langle -q^6;q^{24}\rangle\langle -q^{54};q^{120}\rangle - q^4\langle -q^2;q^{24}\rangle\langle -q^{34};q^{120}\rangle = 0.\end{aligned}$$

This also follows from Schröter's identity (3.1), the only difference being that this time we set $b = -q^2$, before replacing q with q^2. □

Winquist's identity may also be derived from Theorem 2.1. As with the proof of the septuple product identity, our proof needs several applications of Schröter's identity (3.1), rather than following directly from the theorem. We prove Winquist's identity in the following form.

Corollary 4.5. *(Winquist's Identity) Let a and b be non-zero complex numbers and q a complex number with $|q| < 1$. Then*

$$\begin{aligned}\left\langle a;q^2\right\rangle_\infty \left\langle b;q^2\right\rangle_\infty \left\langle ab;q^2\right\rangle_\infty \left\langle \frac{a}{b};q^2\right\rangle_\infty \\ = (q^2,q^2;q^2)_\infty \Big[\left\langle a^3;q^6\right\rangle_\infty \left\{\left\langle b^3q^2;q^6\right\rangle_\infty - b\left\langle b^3q^4;q^6\right\rangle_\infty\right\} \\ - \frac{a}{b}\left\langle b^3;q^6\right\rangle_\infty \left\{\left\langle a^3q^2;q^6\right\rangle_\infty - a\left\langle a^3q^4;q^6\right\rangle_\infty\right\}\Big]. \quad (4.11)\end{aligned}$$

Proof. In (2.3), set $k = 3$, $n_1 = n_2 = 1 = n_3 = N = 1$ (so that $u_1 = u_2 = u_3 = 1$, $v_1 = 1, v_2 = 2$, $v_3 = 3$, $w_1 = 2$, $w_2 = 3$ and $w_3 = 4$), $z = 1$, $a_1 = -a/q$, $a_2 = -ab/q$, $a_3 = -b/q$ and $a = -a/(bq)$. Then the left side of (2.3) becomes the left side of (4.11), and the right side of (2.3) becomes, after some slight manipulation,

$$\begin{aligned}\sum_{j_1=0}^{1}\sum_{j_2=0}^{2} q^{2j_1^2+2j_2^2-2j_1j_2}\left(\frac{1}{b}\right)^{j_1} a^{j_2}\left\langle -bq^{2+2j_1};q^4\right\rangle_\infty \left\langle \frac{-q^{6+6j_1-4j_2}}{b^3};q^{12}\right\rangle_\infty \\ \times \sum_{j_3=0}^{3}\left(\frac{-b}{q^{2j_2+1}}\right)^{j_3} q^{j_3^2}\left\langle -q^{12+6j_3}\frac{b^3}{q^{8j_2}a^3};q^{24}\right\rangle_\infty \left\langle -q^{2j_3}a^3b;q^8\right\rangle_\infty.\end{aligned}$$

Next, we apply Schröter's identity (3.1), with $n_1 = 1$, $n_2 = 3$, a replaced with $-b/q^{2j_2+1}$ and b with $-q^{3-2j_2}/a^3$ to get that the inner sum over j_3 is $\left\langle b/q^{2j_2};q^2\right\rangle_\infty \left\langle q^{6-2j_2}/a^3;q^6\right\rangle_\infty$. Thus the sum above is equal to

$$\begin{aligned}\sum_{j_1=0}^{1}\sum_{j_2=0}^{2} q^{2j_1^2+2j_2^2-2j_1j_2}\left(\frac{1}{b}\right)^{j_1} a^{j_2}\left\langle -bq^{2+2j_1};q^4\right\rangle_\infty \left\langle \frac{-q^{6+6j_1-4j_2}}{b^3};q^{12}\right\rangle_\infty \\ \times \left\langle q^{2j_2}a^3;q^6\right\rangle_\infty \left\langle bq^{-2j_2};q^2\right\rangle_\infty.\end{aligned}$$

By comparison with the right side of (4.11), the result will follow if the next three identities hold:

$$\begin{aligned}&(q^2,q^2;q^2)_\infty\{\left\langle b^3q^2;q^6\right\rangle - b\left\langle b^3q^4;q^6\right\rangle\} \qquad (4.12)\\ &\quad = \left\langle b;q^2\right\rangle \left\{\left\langle -bq^2;q^4\right\rangle \left\langle -b^3q^6;q^{12}\right\rangle + \frac{q^2}{b}\left\langle -bq^4;q^4\right\rangle \left\langle -b^3;q^{12}\right\rangle\right\},\\ &(q^2,q^2;q^2)_\infty \left\langle b^3;q^6\right\rangle \\ &\quad = \left\langle b;q^2\right\rangle \left\{b^2\left\langle -bq^2;q^4\right\rangle \left\langle -b^3q^{10};q^{12}\right\rangle + b\left\langle -bq^4;q^4\right\rangle \left\langle -b^3q^4;q^{12}\right\rangle\right\},\\ &\quad = \left\langle b;q^2\right\rangle \left\{\left\langle -bq^2;q^4\right\rangle \left\langle -b^3q^2;q^{12}\right\rangle + b^2\left\langle -bq^4;q^4\right\rangle \left\langle -b^3q^8;q^{12}\right\rangle\right\}.\end{aligned}$$

We apply Schröter's identity (3.1) again, with $n_1 = 1$, $n_2 = 2$, a replaced with $-b/q$ and, respectively, b kept as b and replaced with bq^2, to get that

$$\begin{aligned}\langle b;q^2\rangle\langle -bq^2;q^4\rangle &= (q^2;q^2)_\infty\left\{\langle -b^3q^4;q^{12}\rangle - b\langle -b^3q^8;q^{12}\rangle\right\},\\ \langle b;q^2\rangle\langle -bq^4;q^4\rangle &= (q^2;q^2)_\infty\left\{b^{-1}\langle -b^3q^2;q^{12}\rangle - b\langle -b^3q^{10};q^{12}\rangle\right\}.\end{aligned}$$

After inserting the expressions on the right above in equation (4.11), the proof of Winquist's identity will follow if it can be shown that the next three identities hold:

$$(q^2;q^2)_\infty\langle b^3;q^6\rangle = \langle -b^3q^4;q^{12}\rangle\langle -b^3q^2;q^{12}\rangle - b^3\langle -b^3q^8;q^{12}\rangle\langle -b^3q^{10};q^{12}\rangle, \tag{4.13}$$

$$(q^2;q^2)_\infty\langle b^3q^2;q^6\rangle = \langle -b^3q^4;q^{12}\rangle\langle -b^3q^6;q^{12}\rangle - q^2\langle -b^3q^{10};q^{12}\rangle\langle -b^3;q^{12}\rangle, \tag{4.14}$$

$$(q^2;q^2)_\infty\langle b^3q^4;q^6\rangle = \langle -b^3q^8;q^{12}\rangle\langle -b^3q^6;q^{12}\rangle - \frac{q^2}{b^3}\langle -b^3q^2;q^{12}\rangle\langle -b^3;q^{12}\rangle. \tag{4.15}$$

Once again appealing to Schröter's identity (3.1), with $n_1 = 1$, $n_2 = 1$ and q replaced with q^3, we get that

$$\begin{aligned}&\langle -aq^3;q^6\rangle\langle -bq^3;q^6\rangle\\ &\quad= \left\langle -\frac{a}{b}q^6;q^{12}\right\rangle\langle -abq^6;q^{12}\rangle + aq^3\left\langle -\frac{a}{b}q^{12};q^{12}\right\rangle\langle -abq^{12};q^{12}\rangle.\end{aligned} \tag{4.16}$$

Identities (4.13), (4.14) and (4.15) follow upon replacing (a,b) in identity (4.16) by, respectively, $(-b^3/q^3,-q)$, $(-1/q,-b^3/q)$ and $(-1/(b^3q),-q)$. This completes the proof of Winquist's identity. □

It is also possible to use identity (2.3) to derive an expression for $(q;q)_\infty^k$ that is different from that given in Corollary 4.2.

Corollary 4.6. *Let $k \geq 3$ be an integer, let $\omega = \exp(2\pi i/k)$ and suppose $|q| < 1$. Then*

$$\begin{aligned}&(q;q)_\infty^k =\\ &\left(q^k;q^k\right)_\infty \sum_{j_1=0}^{1}\sum_{j_2=0}^{2}\cdots\sum_{j_{k-2}=0}^{k-2} q^{(j_1^2+j_2^2+\cdots+j_{k-2}^2)-(j_1j_2+j_2j_3+\cdots+j_{k-3}j_{k-2})}\\ &\times\omega^{-j_1-j_2-j_3-\cdots-j_{k-2}}\left\langle -q^{1+j_1}\omega;q^2\right\rangle_\infty\left\langle (-1)^k q^{k(k-1)/2+kj_{k-2}};q^{k(k-1)}\right\rangle_\infty\\ &\qquad\times\prod_{i=2}^{k-2}\left\langle -q^{i(i+1)/2+(i+1)j_{i-1}-ij_i}\omega^{-i(i+1)/2};q^{i(i+1)}\right\rangle_\infty.\end{aligned} \tag{4.17}$$

Proof. In (4.3) replace z with $-z$ and set $a_i = \omega^i$, $1 \leq i \leq k$, so that the left side becomes $(q^kz^k, q^k/z^k;q^{2k})_\infty\left(q^2;q^2\right)_\infty^k$. Since all the powers of z on the left side have exponent $\equiv 0(\text{mod } k)$, each of the multiple sums on the right side with j_{k-1} fixed, $1 \leq j_{k-1} \leq k-1$, are identically zero, so that the only non-zero sum is the one with $j_{k-1}=0$. With the given values for the a_i, it is clear that $a_1/a_k=\omega$, each $a_i/a_{i+1}=1/\omega$, $a_1a_2\dots a_k=(-1)^{k-1}$ and

$$\frac{a_k a_1 a_2 \dots a_{i-1}}{a_i^i}=\omega^{-i(i+1)/2}.$$

The result follows after cancelling the $(q^kz^k, q^k/z^k; q^{2k})_\infty$ factor on each side, separating off the $k-1$ term in the sum on the right side of the equation that follows from (4.3), and finally replacing q^2 with q. □

It is also possible to derive expressions for $(q;q)_\infty$ as combinations of Jacobi triple products from (2.3).

Corollary 4.7. *Let $k\geq 3$ be an integer, suppose $|q|<1$. Then*

$$(q;q)_\infty=\frac{1}{(q^{2k+1};q^{2k+1})_\infty^{k-1}}\sum_{j_1=0}^{1}\sum_{j_2=0}^{2}\cdots\sum_{j_{k-1}=0}^{k-1}(-1)^{j_{k-1}}$$
$$\times q^{(2k+1)(j_1^2+(j_2-j_1)^2+\cdots+(j_{k-1}-j_{k-2})^2)/2-j_1-j_2-\cdots-j_{k-2}+(k-3/2)j_{k-1}}$$
$$\times\left\langle -q^{(2k+1)(1+j_1)-k+1};q^{2(2k+1)}\right\rangle_\infty\left\langle(-1)^{k+1}q^{k(3k+1)/2+(2k+1)j_{k-1}};q^{(2k+1)k}\right\rangle_\infty$$
$$\times\prod_{i=2}^{k-1}\left\langle -q^{k(i(i+1)+1)+(2k+1)((i+1)j_{i-1}-ij_i)};q^{(2k+1)i(i+1)}\right\rangle_\infty. \quad (4.18)$$

Proof. In (4.3), replace q with $q^{(2k+1)/2}$ and set $z=-1$, $a_1=q^{1/2}$, $a_2=q^{3/2}$, ..., $a_k=q^{k-1/2}$. The left side of the identity then becomes $(q;q)_\infty$ $(q^{2k+1};q^{2k+1})_\infty^{k-1}$, and after some simple algebra on the resulting right side of (4.3), the result follows after dividing both sides of this new expression by the factor $(q^{2k+1};q^{2k+1})_\infty^{k-1}$. □

In a similar vein the following identity holds.

Corollary 4.8. *Let $k\geq 3$ be an integer, suppose $|q|<1$. Then*

$$(q;q)_\infty=\frac{1}{(q^k;q^k)_\infty\,(q^{2k};q^{2k})_\infty^{k-2}}\sum_{j_1=0}^{1}\sum_{j_2=0}^{2}\cdots\sum_{j_{k-1}=0}^{k-1}(-1)^{j_{k-1}}$$
$$\times q^{k(j_1^2+(j_2-j_1)^2+\cdots+(j_{k-1}-j_{k-2})^2)-j_1-j_2-\cdots-j_{k-2}+(k-1)j_{k-1}}$$
$$\times\left\langle -q^{2k(1+j_1)+1};q^{4k}\right\rangle_\infty\left\langle(-1)^{k+1}q^{k(3k-1)/2+2kj_{k-1}};q^{2k^2}\right\rangle_\infty$$
$$\times\prod_{i=2}^{k-1}\left\langle -q^{(2k-1)i(i+1)/2+2k((i+1)j_{i-1}-ij_i)};q^{2ki(i+1)}\right\rangle_\infty. \quad (4.19)$$

Proof. This time in (4.3), replace q with q^k and set $z=-1$, $a_1=q$, $a_2=q^2$, ..., $a_{k-1}=q^{k-1}$, $a_k=1$. The left side of the identity then becomes $(q;q)_\infty$ $(q^k;q^k)_\infty$ $(q^{2k};q^{2k})_\infty^{k-2}$, and the result follows after dividing both sides by $(q^k;q^k)_\infty(q^{2k};q^{2k})_\infty^{k-2}$. □

5. Concluding Remarks

Theorem 2.1 has the restriction that each n_i satisfies $n_i|N$. By using Theorem 2.1 in Cao's paper [3], it is possible to drop this restriction and derive an expansion of a product of an arbitrary number of Jacobi triple products in terms of sums of products of other Jacobi triple products. However, it would be hard to find a general formula such as (2.3) above.

References

[1] Andrews, G. E. *q-series: their development and application in analysis, number theory, combinatorics, physics, and computer algebra.* CBMS Regional Conference Series in Mathematics, **66**. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986. xii+130 pp. ISBN: 0-8218-0716-1.

[2] Borwein, J. M. and Borwein, P. B. *Pi and the AGM. A study in analytic number theory and computational complexity.* Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1987. xvi+414 pp.

[3] Cao, Z. *Integer Matrix Exact Covering Systems and Product Identities for Theta Functions.* Int. Math. Res. Not. no. **19** (2011), 4471–4514.

[4] Cooper, S. *The quintuple product identity.* Int. J. Number Theory **2** (2006), no. 1, 115-161.

[5] Farkas, H. M. and Kra, I. *On the Quintuple Product Identity.* Proc. Amer. Math. Soc. **127** (1999), 771–778.

[6] Gasper, G. and Rahman, M. *Basic hypergeometric series.* With a foreword by Richard Askey. Second edition. Encyclopedia of Mathematics and its Applications, 96. Cambridge University Press, Cambridge, 2004. xxvi+428 pp.

[7] Hirschhorn, M. D. *A simple proof of an identity of Ramanujan*, J. Austral. Math. Soc. Ser. A **34** (1983), 31-35.

[8] Hirschhorn, M. D. *A generalisation of the quintuple product identity.* J. Austral. Math. Soc., Series A **44** (1988) 42-45.

[9] Jacobi, C. G. J. *Fundamenta Nova Theoriae Functionum Ellipticarum.* Knigsberg, Germany: Regiomonti, Sumtibus fratrum Borntraeger, p. 90, 1829.

[10] Winquist, L. *An elementary proof of* $p(11n + 6) \equiv 0 (\mathrm{mod}\ 11)$, J. Combin. Theory **6** (1969), 56-59.

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