What is Mathematics?

West Chester University Mathematics Colloquium,

.

James Mc Laughlin

West Chester University, PA

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Overview

1. Purposes of the Talk
2. What is Mathematics?
3. The Extent of Modern Mathematics
4. Learn to Program Using a Computer Algebra System such as *Mathematica* or Maple
5. The Mathematical areas in which I do research
6. Interlude: qf_1^{24}
7. Connection to the work on Vanishing Coefficients
8. Chebyshev polynomials of the Second Kind
9. Interlude: Some Useful Online Mathematical Resources
10. Properties of Chebyshev polynomials of the second kind
11. Applications to the Fourier Coefficients of Hecke Eigenforms

Alternative Titles

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"Beware The WHAT IS MATHEMATICS? Scam"

Purposes of the Talk

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3. To demonstrate how most mathematicians work in a very localized area of mathematics.

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3. To demonstrate how most mathematicians work in a very localized area of mathematics.
4. To provide an overview of the particular area of mathematics in which I do research.

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1. To attempt to provide a picture for non-mathematicians of the kinds of things mathematicians spend their time on.
2. To attempt to provide a picture for students thinking of pursuing a career in mathematics of how mathematicians spend their time.
3. To demonstrate how most mathematicians work in a very localized area of mathematics.
4. To provide an overview of the particular area of mathematics in which I do research.
5. To indicate the importance of computer algebra systems to my own research.

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Useful real-world applications of an area of mathematics may only come many years after the theory has been worked out (if at all).

What is Mathematics?

What is Mathematics?

A (Very) Little Philosophy of Mathematics I

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Mathematical Realism (Wikipedia)

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Mathematical Universe Hypothesis (Wikipedia)

A (Very) Little Philosophy of Mathematics II

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Observers, including humans, are "self-aware substructures (SASs)".
In any mathematical structure complex enough to contain such substructures, they "will subjectively perceive themselves as existing in a physically 'real' world."

A (Very) Little Philosophy of Mathematics III

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A (Very) Little Philosophy of Mathematics IV

Gödel's incompleteness theorems (Wikipedia)

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' For any such consistent formal system, there will always be statements about natural numbers that are true, but that are unprovable within the system.

' The second incompleteness theorem, an extension of the first, shows that the system cannot demonstrate its own consistency."

The Extent of Modern Mathematics

Mathematics Subject Classification

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At the top level, 64 mathematical disciplines are labeled with a unique two-digit number.

Mathematics Subject Classification - First-level areas I

Mathematics Subject Classification - First-level areas I

General/foundations

Mathematics Subject Classification - First-level areas I

General/foundations

1. 00: General (Includes topics such as recreational mathematics, philosophy of mathematics and mathematical modeling.)

Mathematics Subject Classification - First-level areas I

General/foundations

1. 00: General (Includes topics such as recreational mathematics, philosophy of mathematics and mathematical modeling.)
2. 01: History and biography

Mathematics Subject Classification - First-level areas I

General/foundations

1. 00: General (Includes topics such as recreational mathematics, philosophy of mathematics and mathematical modeling.)
2. 01: History and biography
3. 03: Mathematical logic and foundations, including model theory. computability theory. set theory. proof theory. and algebraic logic

Mathematics Subject Classification - First-level areas II

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Discrete mathematics/algebra

Discrete mathematics/algebra

1. 05: Combinatorics

Mathematics Subject Classification - First-level areas II

Discrete mathematics/algebra

1. 05: Combinatorics
2. 06: Order theory

Mathematics Subject Classification - First-level areas II

Discrete mathematics/algebra

1. 05: Combinatorics
2. 06: Order theory
3. 08: General algebraic systems

Mathematics Subject Classification - First-level areas II

Discrete mathematics/algebra

1. 05: Combinatorics
2. 06: Order theory
3. 08: General algebraic systems
4. 11: Number theory

Mathematics Subject Classification - First-level areas II

Discrete mathematics/algebra

1. 05: Combinatorics
2. 06: Order theory
3. 08: General algebraic systems
4. 11: Number theory
5. 12: Field theory and polynomials

Mathematics Subject Classification - First-level areas II

Discrete mathematics/algebra

1. 05: Combinatorics
2. 06: Order theory
3. 08: General algebraic systems
4. 11: Number theory
5. 12: Field theory and polynomials
6. 13: Commutative rings and algebras

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2. 06: Order theory
3. 08: General algebraic systems
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5. 12: Field theory and polynomials
6. 13: Commutative rings and algebras
7. 14: Algebraic geometry

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8. 15: linear and multillnear algebra: matrix theory

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9. 16: Associative rings and associative algebras

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10. 17: Non-associative rings and non-associative algebras

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9. 16: Associative rings and associative algebras
10. 17: Non-associative rings and non-associative algebras
11. 18': Category theory: homological algebra

Mathematics Subject Classification - First-level areas II

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9. 16: Associative rings and associative algebras
10. 17: Non-associative rings and non-associative algebras
11. 18': Category theory: homological algebra
12. 19: K-theory

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Discrete mathematics/algebra

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9. 16: Associative rings and associative algebras
10. 17: Non-associative rings and non-associative algebras
11. 18': Category theory: homological algebra
12. 19: K-theory
13. 20: Group theory and generalizations

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Discrete mathematics/algebra

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8. 15: linear and multillnear algebra: matrix theory
9. 16: Associative rings and associative algebras
10. 17: Non-associative rings and non-associative algebras
11. 18': Category theory: homological algebra
12. 19: K-theory
13. 20: Group theory and generalizations
14. 22: Topological groups, Lie groups, and analysis upon them

Mathematics Subject Classification - First-level areas III

Mathematics Subject Classification - First-level areas III

Analysis

Mathematics Subject Classification - First-level areas III

Analysis

1. 26: Real functions, including derivatives and integrals

Mathematics Subject Classification - First-level areas III

Analysis

1. 26: Real functions, including derivatives and integrals
2. 28: Measure and integration

Mathematics Subject Classification - First-level areas III

Analysis

1. 26: Real functions, including derivatives and integrals
2. 28: Measure and integration
3. 30: Complex functions, including approximation theory in the complex domain

Mathematics Subject Classification - First-level areas III

Analysis

1. 26: Real functions, including derivatives and integrals
2. 28: Measure and integration
3. 30: Complex functions, including approximation theory in the complex domain
4. 31: Potential theory

Mathematics Subject Classification - First-level areas III

Analysis

1. 26: Real functions, including derivatives and integrals
2. 28: Measure and integration
3. 30: Complex functions, including approximation theory in the complex domain
4. 31: Potential theory
5. 32: Several complex variables and analytic spaces

Mathematics Subject Classification - First-level areas III

Analysis

1. 26: Real functions, including derivatives and integrals
2. 28: Measure and integration
3. 30: Complex functions, including approximation theory in the complex domain
4. 31: Potential theory
5. 32: Several complex variables and analytic spaces
6. 33: Special functions

Mathematics Subject Classification - First-level areas III

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3. 30: Complex functions, including approximation theory in the complex domain
4. 31: Potential theory
5. 32: Several complex variables and analytic spaces
6. 33: Special functions
7. 34: Ordinary differential equations

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7. 34: Ordinary differential equations
8. 35: Partial differential equations

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7. 34: Ordinary differential equations
8. 35: Partial differential equations
9. 37: Dynamical systems and ergodic theory

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8. 35: Partial differential equations
9. 37: Dynamical systems and ergodic theory
10. 39: Difference equations and functional equations

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Analysis

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5. 32: Several complex variables and analytic spaces
6. 33: Special functions
7. 34: Ordinary differential equations
8. 35: Partial differential equations
9. 37: Dynamical systems and ergodic theory
10. 39: Difference equations and functional equations
11. 40: Sequences, series, summability

Mathematics Subject Classification - First-level areas IV

Mathematics Subject Classification - First-level areas IV

Analysis, continued

Mathematics Subject Classification - First-level areas IV

Analysis, continued

1. 41: Approximations and expansions

Mathematics Subject Classification - First-level areas IV

Analysis, continued

1. 41: Approximations and expansions
2. 42: Harmonic analysis, including Fourier analysis, Fourier transforms, trigonometric approximation, trigonometric interpolation, and orthogonal functions

Mathematics Subject Classification - First-level areas IV

Analysis, continued

1. 41: Approximations and expansions
2. 42: Harmonic analysis, including Fourier analysis, Fourier transforms, trigonometric approximation, trigonometric interpolation, and orthogonal functions
3. 43: Abstract harmonic analysis

Mathematics Subject Classification - First-level areas IV

Analysis, continued

1. 41: Approximations and expansions
2. 42: Harmonic analysis, including Fourier analysis, Fourier transforms, trigonometric approximation, trigonometric interpolation, and orthogonal functions
3. 43: Abstract harmonic analysis
4. 44: Integral transforms, operational calculus

Mathematics Subject Classification - First-level areas IV

Analysis, continued

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2. 42: Harmonic analysis, including Fourier analysis, Fourier transforms, trigonometric approximation, trigonometric interpolation, and orthogonal functions
3. 43: Abstract harmonic analysis
4. 44: Integral transforms, operational calculus
5. 45: Integral equations

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3. 43: Abstract harmonic analysis
4. 44: Integral transforms, operational calculus
5. 45: Integral equations
6. 46: Functional analysis, including infinite-dimensional holomorphy, integral transforms in distribution spaces

Mathematics Subject Classification - First-level areas IV

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7. 47: Operator theory

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5. 45: Integral equations
6. 46: Functional analysis, including infinite-dimensional holomorphy, integral transforms in distribution spaces
7. 47: Operator theory
8. 49: Calculus of variations and optimal control: optimization (including geometric integration theory)

Mathematics Subject Classification - First-level areas V

Mathematics Subject Classification - First-level areas V

Geometry and topology

Mathematics Subject Classification - First-level areas V

Geometry and topology

1. 51: Geometry

Mathematics Subject Classification - First-level areas V

Geometry and topology

1. 51: Geometry
2. 52: Convex geometry and discrete geometry

Mathematics Subject Classification - First-level areas V

Geometry and topology

1. 51: Geometry
2. 52: Convex geometry and discrete geometry
3. 53: Differential geometry

Mathematics Subject Classification - First-level areas V

Geometry and topology

1. 51: Geometry
2. 52: Convex geometry and discrete geometry
3. 53: Differential geometry
4. 54: General topology

Mathematics Subject Classification - First-level areas V

Geometry and topology

1. 51: Geometry
2. 52: Convex geometry and discrete geometry
3. 53: Differential geometry
4. 54: General topology
5. 55: Algebraic topology

Mathematics Subject Classification - First-level areas V

Geometry and topology

1. 51: Geometry
2. 52: Convex geometry and discrete geometry
3. 53: Differential geometry
4. 54: General topology
5. 55: Algebraic topology
6. 57: Manifolds

Mathematics Subject Classification - First-level areas V

Geometry and topology

1. 51: Geometry
2. 52: Convex geometry and discrete geometry
3. 53: Differential geometry
4. 54: General topology
5. 55: Algebraic topology
6. 57: Manifolds
7. 58: Global analysis, analysis on manifolds (Including Infinite-dimensional holomorphy)

Mathematics Subject Classification - First-level areas VI

Applied mathematics / other

Mathematics Subject Classification - First-level areas VI

Applied mathematics / other

1. 60 Probability theory and stochastic processes

Mathematics Subject Classification - First-level areas VI

Applied mathematics / other

1. 60 Probability theory and stochastic processes
2. 62 Statistics

Mathematics Subject Classification - First-level areas VI

Applied mathematics / other

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2. 62 Statistics
3. 65 Numerical analysis

Mathematics Subject Classification - First-level areas VI

Applied mathematics / other

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2. 62 Statistics
3. 65 Numerical analysis
4. 68 Computer science

Mathematics Subject Classification - First-level areas VI

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Applied mathematics / other

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3. 65 Numerical analysis
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6. 74 Mechanics of deformable solids

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3. 65 Numerical analysis
4. 68 Computer science
5. 70 Mechanics (Including particle mechanics)
6. 74 Mechanics of deformable solids
7. 76 Fluid mechanics

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7. 76 Fluid mechanics
8. 78 Optics, electromagnetic theory

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10. 81 Quantum theory
11. 82 Statistical mechanics, structure of matter

Mathematics Subject Classification - First-level areas VII

Mathematics Subject Classification - First-level areas VII

Applied mathematics / other, continued

Applied mathematics / other, continued

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Mathematics Subject Classification - First-level areas VII

Applied mathematics / other, continued

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Mathematics Subject Classification - First-level areas VII

Applied mathematics / other, continued

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3. 86 Geophysics

Mathematics Subject Classification - First-level areas VII

Applied mathematics / other, continued

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4. 90 Operations research, mathematical programming

Mathematics Subject Classification - First-level areas VII

Applied mathematics / other, continued

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Applied mathematics / other, continued

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7. 93 Systems theory; control, Including optimal control
8. 94 Information and communication, circuits
9. 97 Mathematics education

The Complete MSC

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Most Mathematics PhD granting institutions have some program to connect students with an area of mathematics that matches their interests.

Learn to Program Using a Computer Algebra System such as *Mathematica* or Maple

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- figuring out how to prove any new discoveries you might make is fun
- collaborating with other people on math projects is fun

An Example from my own Research - q-products and q-series

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$$\text{For } |q| < 1, \qquad (q;q)_\infty := (1-q)(1-q^2)(1-q^3)\cdots$$

$$f_1 := (q;q)_\infty \qquad f_j := (q^j;q^j)_\infty$$

The series expansion for f_1:

$$\begin{aligned} f_1 = (q;q)_\infty = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} \\ - q^{35} - q^{40} + q^{51} + q^{57} - q^{70} - q^{77} + q^{92} + q^{100} \\ - q^{117} - q^{126} + q^{145} + q^{155} - q^{176} - q^{187} \ldots \end{aligned}$$

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Notice that the coefficients of most powers of q are zero.

q-products Continued

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The list of coefficients:

q-products Continued

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$$1, -1, -1, 0, 0, 1, 0, 1, 0, 0, 0, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0,$$
$$0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0,$$
$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$
$$0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0,$$
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q-products Continued

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The series $\sum_{n=0}^{\infty} c(n)q^n$ is *lacunary* if

$$\lim_{x\to\infty} \frac{|\{0 \leq n \leq x \mid c(n) = 0\}|}{x} = 1.$$

q-products Continued

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q-products Continued

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Serre: for even positive integers s, f_1^s is lacunary if and only if

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For odd positive integers s it is known that f_1^s lacunary for $s = 1$ and $s = 3$, but nothing that is conclusive is known.

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An *eta quotient* is a finite product of the form $\prod_j f_j^{n_j}$, for some integers $j \in \mathbb{N}$ and $n_j \in \mathbb{Z}$.

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One could also ask questions about which of these more general eta quotients are lacunary.

A Result of Han and Ono

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Define the sequences $\{a(n)\}$ and $\{b(n)\}$ by

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Theorem

(*Han and Ono, 2011*)

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Moreover, we have that $a(n) = b(n) = 0$ *precisely for those non-negative* n *for which* $3n + 1$ *has a prime factor* p *of the form* $p = 3k + 2$ *with odd exponent for some integer* k.

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In other words, if the prime factorization of $3n + 1$ has the form $3n + 1 = \ldots p^{2r+1} \ldots$ for some integer $r \geq 0$, then $a(n) = b(n) = 0$, and $a(n) \neq 0$, $b(n) \neq 0$ otherwise.

The Result of Han and Ono in More Detail

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$$f_1^8 = 1 - 8q + 20q^2 - 70q^4 + 64q^5 + 56q^6 - 125q^8 - 160q^9 + 308q^{10} + 110q^{12} - 520q^{14} + 57q^{16} + 560q^{17} + 182q^{20} + \dots,$$

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Notice that the two series vanish for the same powers of q, namely q^n with $n = 3, 7, 11, 13, 15, 18, 19 \dots.$

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Further, for any n in this list, $3n+1$ has a prime factor p of the form $p = 3k+2$ with odd exponent.

The Result of Han and Ono in More Detail

$$f_1^8 = 1 - 8q + 20q^2 - 70q^4 + 64q^5 + 56q^6 - 125q^8 - 160q^9 + 308q^{10} + 110q^{12} - 520q^{14} + 57q^{16} + 560q^{17} + 182q^{20} + \dots,$$

$$\frac{f_3^3}{f_1} = 1 + q + 2q^2 + 2q^4 + q^5 + 2q^6 + q^8 + 2q^9 + 2q^{10} + 2q^{12} + 2q^{14} + 3q^{16} + 2q^{17} + 2q^{20} + \dots.$$

Notice that the two series vanish for the same powers of q, namely q^n with $n = 3, 7, 11, 13, 15, 18, 19 \dots.$

Further, for any n in this list, $3n + 1$ has a prime factor p of the form $p = 3k + 2$ with odd exponent.

(For example, for $n = 11$, $3n + 1 = 3(11) + 1 = 34 = 2(17^1)$ and $17 = 3(5) + 2$.)

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Theorem 1 motivated the speaker to investigate experimentally if similar results held for other pairs of eta quotients.

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Theorem 1 motivated the speaker to investigate experimentally if similar results held for other pairs of eta quotients.

This was done using some simple *Mathematica* programs (next slides).

Some *Mathematica* Code - I

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We search for eta quotients $f_1^t f_i^j f_k^l f_m^n f_r^s f_u^v$ with coefficients that vanish identically with, say, f_1^6

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```
lim = 12; plim = 12;
Clear[i, j, k, l, m, n, r, s, u, v, t,f,q,a]
f[a_]=QPochhammer[q^a, q^a];
pra = f[1]^6;
lsprc = {pra};
lsprd = { };
cla = CoefficientList[Series[pra, {q, 0, 60}], q];
posa = Intersection[Flatten[Position[cla, 0]], Range[50]];
```

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```

The list **posa** contains the positions of the vanishing coefficients in the series expansion of f_1^6, up to q^{50}.

Some *Mathematica* Code - II

Next a set of **For[ ]** loops that the integer variable $i, j, k, l, m, n, r, s, t, u, v$ cycle through to produce the eta quotients $f_1^t f_i^j f_k^l f_m^n f_r^s f_u^v$:

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```
For[t = -plim, t ≤ plim, t++,
  For[j = -plim, j ≤ plim, j++,
   For[i = 2, i ≤ lim - 4, i++,
    For[k = i + 1, k ≤ lim - 3, k++,
     For[l = -plim, l ≤ plim, l++,
      For[m = k + 1, m ≤ lim - 2, m++,
       For[r = m + 1, r ≤ lim - 1, r++,
        For[s = -plim, s ≤ plim, s++,
         For[u = r + 1, u ≤ lim, u++,
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      For[m = k + 1, m ≤ lim - 2, m++,
       For[r = m + 1, r ≤ lim - 1, r++,
        For[s = -plim, s ≤ plim, s++,
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(what happens inside the "For" loops - next slide)

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      For[m = k + 1, m ≤ lim - 2, m++,
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        For[s = -plim, s ≤ plim, s++,
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```
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n = (6 - i j - k l - r s - t - u v)/m;
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So the eta quotients **prb** gets added to the list **lsprc** if it appears that its coefficients vanish identically with those of f_1^6,

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clb = CoefficientList[Series[prb, q, 0, 60], q];
posb = Intersection[Flatten[Position[clb, 0]], Range[50]];
If[posa == posb, lsprc = Append[lsprc, prb];];
If[SubsetQ[posb, posa] && (! SubsetQ[posa, posb]),
lsprd = Append[lsprd, prb];];
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So the eta quotients **prb** gets added to the list **lsprc** if it appears that its coefficients vanish identically with those of f_1^6, and gets added to the list **lsprd** if it appears that the vanishing coefficients of f_1^6 are a proper subset of the vanishing coefficients of **prb**.

Experimental Results

What was discovered as a result of these computer algebra experiments is summarized as follows.

Other eta quotients with identically vanishing coefficients I

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Let $(A(q), B(q))$ be any of the pairs

$$\left\{ \left(f_1^4, \frac{f_1^8}{f_2^2}\right), \left(f_1^4, \frac{f_1^{10}}{f_3^2}\right), \left(f_1^6, \frac{f_2^4}{f_1^2}\right), \left(f_1^6, \frac{f_1^{14}}{f_2^4}\right), \left(f_1^{10}, \frac{f_2^6}{f_1^2}\right), \left(f_1^{14}, \frac{f_3^5}{f_1}\right), \left(f_1^{14}, \frac{f_2^8}{f_1^2}\right) \right\}. \quad (3)$$

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$$\left. \left(f_1^{10}, \frac{f_2^6}{f_1^2}\right), \left(f_1^{14}, \frac{f_3^5}{f_1}\right), \left(f_1^{14}, \frac{f_2^8}{f_1^2}\right) \right\}. \quad (3)$$

For any such pair $(A(q), B(q))$, define the sequences $\{a(n)\}$ and $\{b(n)\}$ by

$$A(q) =: \sum_{n=0}^{\infty} a(n)q^n, \qquad B(q) =: \sum_{n=0}^{\infty} b(n)q^n. \quad (4)$$

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Then, for each pair, $a(n) = 0 \iff b(n) = 0$, with criteria for when exactly this happens.

Other eta quotients with identically vanishing coefficients II

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For the pairs

$$\left\{ \left(f_1^{26}, \frac{f_3^9}{f_1}\right), \left(f_1^{26}, \frac{f_2^{16}}{f_1^6}\right) \right\} \tag{5}$$

Other eta quotients with identically vanishing coefficients II

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$a(n) = b(n) = 0$ if $12n + 13$ satisfies a criteria of Serre for $a(n) = 0$.

Other eta quotients with identically vanishing coefficients II

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$a(n) = b(n) = 0$ if $12n + 13$ satisfies a criteria of Serre for $a(n) = 0$.
How to prove these results on identically vanishing coefficients?

Aside: other infinite products with vanishing coefficients I

Aside: other infinite products with vanishing coefficients I

Consider

$$\prod_{n=0}^{\infty} \frac{(1-q^{8n+1})(1-q^{8n+7})}{(1-q^{8n+3})(1-q^{8n+5})} = 1 - q + q^3 - q^4 + q^5 - 2q^7 + 2q^8 - q^9$$
$$+ 2q^{11} - 3q^{12} + 2q^{13} - 2q^{15} + 4q^{16} - 4q^{17} + 4q^{19} - 6q^{20} + 5q^{21}$$
$$- 6q^{23} + 9q^{24} - 6q^{25} + 7q^{27} - 12q^{28} + 9q^{29} + \cdots =: \sum_{n=0}^{\infty} a_n q^n$$

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List of coefficients:

$$1, -1, 0, 1, -1, 1, 0, -2, 2, -1, 0, 2, -3, 2, 0, -2, 4, -4, 0, 4, -6, 5,$$
$$0, -6, 9, -6, 0, 7, -12, 9, 0, \ldots$$

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List of coefficients:

$$1, -1, 0, 1, -1, 1, 0, -2, 2, -1, 0, 2, -3, 2, 0, -2, 4, -4, 0, 4, -6, 5, 0, -6, 9, -6, 0, 7, -12, 9, 0, \ldots$$

The list of n such that $a_n = 0$:

$$2, 6, 10, 14, 18, 22, 26, 30, \ldots$$

Aside: other infinite products with vanishing coefficients I

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$$\prod_{n=0}^{\infty} \frac{(1-q^{8n+1})(1-q^{8n+7})}{(1-q^{8n+3})(1-q^{8n+5})} = 1 - q + q^3 - q^4 + q^5 - 2q^7 + 2q^8 - q^9$$
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$$1, -1, 0, 1, -1, 1, 0, -2, 2, -1, 0, 2, -3, 2, 0, -2, 4, -4, 0, 4, -6, 5,$$
$$0, -6, 9, -6, 0, 7, -12, 9, 0, \ldots$$

The list of n such that $a_n = 0$:

$$2, 6, 10, 14, 18, 22, 26, 30, \ldots = \{4n + 2 | n \geq 0\}.$$

Alladi, Andrews, and others have worked on such infinite products.

Aside: other infinite products with vanishing coefficients II

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On the other hand, consider one of the infinite products from a few slides back, f_1^8:

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The list of n such that $b_n = 0$:

$$3, 7, 11, 13, 15, 18, 19, 23, 27, 28, 29, 31, 35, 38, 39, 43, 45, 47, 48, 51, 53, 55, 59, 61, 62, 63, 67, 68, 71, 73, 75, 77, 78, 79, 83, 84, 87, 88, 91, 93, 95, 98, 99, \ldots$$

Other eta quotients with identically vanishing coefficients III

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This research is described more fully in the second talk next week, with most of the remainder of this talk being a description of some more elementary work that arose as a side project to the above work.

The Mathematical areas in which I do research.

Research Areas - Continued Fractions I

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Definition: *continued fractions:*

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$b_0 + K_{n=1}^{\infty} \frac{a_n}{b_n}$ *converges* if the sequence $\{f_n\}$ converges.

Research Areas - Continued Fractions II

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Regular continued fraction expansions:

Research Areas - Continued Fractions II

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Research Areas - Continued Fractions II

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Research Areas - Continued Fractions II

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Research Areas - Continued Fractions II

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Research Areas - Continued Fractions III

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Example:

$$(q^2;q^5)_\infty = (1-q^2)(1-q^7)(1-q^{12})(1-q^{17})\dots.$$

Research Areas - Continued Fractions IV

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$$\lim_{j\to\infty} K_{2j+1}(q) = \frac{1}{K(-1/q)},$$

$$\lim_{j\to\infty} K_{2j}(q) = \frac{K(1/q^4)}{q}.$$

Research Areas - Continued Fractions V

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What if $|q| = 1$ but q is not a root of unity?

This was an open question until my thesis, where I showed the existence of an uncountable set of points (of measure 0) for which $K(q)$ diverges.

Research Areas - Continued Fractions VI

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Research Areas - Continued Fractions VI

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Research Areas - Continued Fractions VII

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Research Areas - Continued Fractions VII

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Research Areas - Continued Fractions VII

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Komatsu:

$$[0, \overline{a^k}]_{k=1}^{\infty} := [0; a, a^2, a^3, a^4, \cdots = \frac{\sum_{s=0}^{\infty} a^{-(s+1)^2} \prod_{i=1}^{s} (a^{2i}-1)^{-1}}{\sum_{s=0}^{\infty} a^{-s^2} \prod_{i=1}^{s} (a^{2i}-1)^{-1}}.$$

Research Areas - Continued Fractions VIII

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Research Areas - Continued Fractions VIII

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Research Areas - Continued Fractions IX

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Research Areas - Continued Fractions IX

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Research Areas - Continued Fractions IX

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where $a_i(x) \in \mathbb{Z}[x]$ for $i \geq 1$ (a *specializable* continued fraction expansion). By letting x be an integer, he got expansions such as:

$$\sum_{n \geq 0} \frac{1}{T_{4^n}(2)} = [0; 1, 1, 23, 1, 2, 1, 18815, 3, 1, 23, 3, 1, 23, 1, 2, 1, 106597754640383, 3, 1, 23, 1, 3, 23, 1, 3, 18815, 1, 2, 1, 23, 3, 1, 23, \cdots],$$

$T_l(x)$ being the l-th Chebyshev polynomial of the first kind.

Research Areas - Continued Fractions X

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Research Areas - Continued Fractions X

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likewise has a specializable continued fraction expansion.
Specialization likewise led to results such as

$$\prod_{j=0}^{\infty}\left(1+\frac{1}{T_{6^j}(3)}\right) =$$

$$[1; 2, 1, 1632, 1, 2, 1, 3542435884041835200, 1, 2, 1, 1632, 1, 2, 1,$$
$$2602953921777123453854421658848856619640265580447716525 3$$
$$93363412220776182840684687324960468372004114475959136 00,$$
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Research Areas - Integer Partitions I

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Research Areas - Integer Partitions II

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We recall also that

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{\nu+2m}}{m!\Gamma(\nu+m+1)}$$

denotes the modified Bessel function of the first kind.

Research Areas - Integer Partitions III

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Theorem

(Rademacher) If n is a positive integer, then

$$p(n) = \frac{2\pi}{(24n-1)^{3/4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right). \qquad (8)$$

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For example,

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and yet six terms of the series are sufficient to get within 0.5 of $p(500)$.
The idea of course is that if a partial sum is known to be within 0.5 of the value of the series, then the nearest integer gives the exact value of $p(n)$.

Research Areas - Integer Partitions IV

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Mc L. and Parsell (2012)

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$$R := \frac{(r-1)(s-1)}{24}, \qquad \delta_k := \frac{(r/r_k - r_k)(s/s_k - s_k)}{24}. \tag{9}$$

Research Areas - Integer Partitions V

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$$\frac{F\left(\frac{H_{h,k}}{k} + \frac{i}{z}\right) F\left(\frac{H_{hrs/(r_k s_k),k/(r_k s_k)}}{k/(r_k s_k)} + \frac{i r_k^2 s_k^2}{rsz}\right)}{F\left(\frac{H_{hr/r_k,k/r_k}}{k/r_k} + \frac{i r_k^2}{rz}\right) F\left(\frac{H_{hs/s_k,k/s_k}}{k/s_k} + \frac{i s_k^2}{sz}\right)} := \sum_{m=0}^{\infty} c_m(h,k) \exp\left(\frac{-2\pi m r_k s_k}{rsz}\right).$$

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If $n > R$, then

$$p_{r,s}(n) = \sum_{k=1}^{\infty} \sum_{m=0}^{\lfloor \delta_k \rfloor} \frac{2\pi A_{k,m}(n)}{k} \sqrt{\frac{r_k s_k(\delta_k - m)}{rs(n-R)}}\, I_1\left(\frac{4\pi}{k}\sqrt{\frac{r_k s_k}{rs}(\delta_k - m)(n-R)}\right),$$

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Research Areas - Integer Partitions VI

As an example, we consider the convergence of the sum of the series to

$$p_{14,15}(500) = 310,093,947,025,073,675,623,$$

Research Areas - Integer Partitions VI

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by examining the difference $p_{14,15}(500) - S_N$, where S_N is the Nth partial sum of the series.

Research Areas - Integer Partitions VII

Research Areas - Integer Partitions VII

N	S_N	$p_{14,15}(500) - S_N$
1	31009394702504993 2429.8505	-2.374319315×10^7
2	310093947025073675628.9283	5.9283
3	310093947025073675414.3591	-208.6409
4	310093947025073675623.3258	0.3258
5	310093947025073675623.3258	0.3258
6	310093947025073675623.3723	0.3723
7	310093947025073675623.3723	0.3723
8	310093947025073675623.3723	0.3723
9	310093947025073675623.2793	0.2793
10	310093947025073675623.2793	0.2793
11	310093947025073675623.4447	0.4447

Table: The fast initial convergence of the series for $p_{14,15}(500)$.

Research Areas - q-Series I

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The "famous" Rogers-Ramanujan identities:

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$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})}, \tag{10}$$

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A good deal of research in basic hypergeometric series involves "infinite series = infinite product" as above, and various ways of producing and proving these.

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Other work involves identities of the type

"infinite series$_1$ = infinite product$\times$ infinite series$_2$".

Research Areas - *q*-Series II

Research Areas - q-Series II

1. Using Bailey pairs (with Doug Bowman and Andrew Sills, 2009)

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q^3;q^3)_{n-1}}{(-q;q)_n(q;q)_{2n-1}}=\frac{1}{(q;q)_\infty}\Big((q^{12},q^{15},q^{27};q^{27})_\infty - 2q^2(-q^{33},-q^{75},q^{108};q^{108})_\infty+2q^7(-q^{15},-q^{93},q^{108};q^{108})_\infty\Big) \quad (12)$$

Research Areas - q-Series III

Research Areas - *q*-Series III

2. Using mock theta functions:

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$$\sum_{r=-\infty}^{\infty}(10r+1)q^{(5r^2+r)/2} = \left(\frac{4q(q^4,q^{16},q^{20};q^{20})_\infty}{(q^2;q^4)_\infty}+\frac{(q^2,q^3,q^5;q^5)_\infty}{(-q;q)_\infty}\right)\frac{(q;q)_\infty^2}{(-q;q)_\infty},$$

$$\sum_{r=-\infty}^{\infty}(10r+3)q^{(5r^2+3r)/2} = \left(\frac{4(q^8,q^{12},q^{20};q^{20})_\infty}{(q^2;q^4)_\infty}-\frac{(q,q^4,q^5;q^5)_\infty}{(-q;q)_\infty}\right)\frac{(q;q)_\infty^2}{(-q;q)_\infty}.$$

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Research Areas - q-Series IV

Research Areas - q-Series IV

3. Continued Fractions

Research Areas - q-Series IV

3. Continued Fractions (again),

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Research Areas - q-Series IV

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Research Areas - q-Series V

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$$\sum_{\vec{m}} \frac{q^{m_1(m_1-m_2)+m_2(m_2-m_3)+\cdots+m_{k-1}(m_{k-1}-m_k)+m_k^2}}{(q;q)_{m_1}(q;q)_{m_1-m_2}\ldots(q;q)_{m_{k-1}}(q;q)_{m_{k-1}-m_k}(q;q)_{m_k}^2} = \frac{1}{(q;q)_\infty^k}; \quad (14)$$

Research Areas - Vanishing Coefficients I

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West Chester University

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Research Areas - Vanishing Coefficients III

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Since $V^{-1} \pmod{59} = 20$, then
$-xV^{-1} = -(7)(20) \equiv 37 \pmod{59}$, so $v = 37$.

Research Areas - Vanishing Coefficients IV

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Hence, by the theorem, if the sequence $\{r_n\}$ is defined by

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Interlude: qf_1^{24} and the Ramanujan τ Function

Interlude - The Importance of Modular Forms

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"There are five elementary arithmetical operations: addition, subtraction, multiplication, division, and . . . modular forms." - Martin Eichler.

Srinivasa Ramanujan Aiyangar

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22 December 1887 - 26 April 1920 (aged 32)

The Ramanujan τ Function

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$$q\prod_{m=1}^{\infty}(1-q^m)^{24} =: \sum_{n=1}^{\infty}\tau(n)q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + 84480q^8 - 113643q^9 - 115920q^{10} + 534612q^{11} - 370944q^{12} - 577738q^{13} + 401856q^{14} + 1217160q^{15} + 987136q^{16} - \dots$$

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For example, with $p = 2$ and $r = 3$,
$\tau(2)\tau(2^3) - 2^{11}\tau(2^2) = (-24)84480 - 2^{11}(-1472)$
$= 987136 = \tau(2^4)$.

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Conjecture: $\tau(n) \neq 0$ for any positive integer n (D.H. Lehmer, 1947).

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$$a_{p^{n+1}} = a_{p^n} a_p - \chi(p) p^{k-1} a_{p^{n-1}}. \tag{18}$$

As with $\tau(p^{n+1})$, the recurrence relation (18) implies that $a_{p^{n+1}}$ is a polynomial in a_p.

It was trying to determine these polynomials that led to results presented later in this talk.

Connection to the work on Vanishing Coefficients

Results involving f_1^{26} Again

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For the pairs

$$\left\{ \left(f_1^{26}, \frac{f_3^9}{f_1}\right), \left(f_1^{26}, \frac{f_2^{16}}{f_1^6}\right) \right\} \tag{19}$$

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While trying to prove the (possibly false) reverse direction, the speaker was led to the result described in the next few slides.

Connection to Chebyshev polynomials of the Second Kind

Chebyshev polynomials of the Second Kind

Chebyshev polynomials of the Second Kind I

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Recall the Chebyshev polynomials of the second kind, $\{U_n(x)\}$,

Chebyshev polynomials of the Second Kind I

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Recall the Chebyshev polynomials of the second kind, $\{U_n(x)\}$, defined by $U_0(x) = 1$, $U_1(x) = 2x$, and the recursive formula

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x). \tag{20}$$

Chebyshev polynomials of the Second Kind II

Chebyshev polynomials of the Second Kind II

The first 10 Chebyshev polynomials of the second kind:

Chebyshev polynomials of the Second Kind II

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$$
\begin{aligned}
U_1(x) &= 2x,\\
U_2(x) &= 4x^2 - 1,\\
U_3(x) &= 8x^3 - 4x,\\
U_4(x) &= 16x^4 - 12x^2 + 1,\\
U_5(x) &= 32x^5 - 32x^3 + 6x,\\
U_6(x) &= 64x^6 - 80x^4 + 24x^2 - 1,\\
U_7(x) &= 128x^7 - 192x^5 + 80x^3 - 8x,\\
U_8(x) &= 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1,\\
U_9(x) &= 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x,\\
U_{10}(x) &= 1024x^{10} - 2304x^8 + 1792x^6 - 560x^4 + 60x^2 - 1,\\
&\vdots
\end{aligned}
$$

A Formula for a_p^n and Chebyshev Polynomials of the Second Kind

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A Formula for a_p^n and Chebyshev Polynomials of the Second Kind

Proposition

Let $f(q) = q + \sum_{n=2}^{\infty} a_n q^n$ be a normalized Hecke eigenform of weight k, level N, and Nebentypus χ. Let $p \nmid N$ be a prime, so that

$$a_{p^{n+1}} = a_{p^n} a_p - \chi(p) p^{k-1} a_{p^{n-1}}. \tag{21}$$

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Then, after fixing a value for $\sqrt{\chi(p)}$*,*

$$a_{p^n} = \left(-p^{(k-1)/2}\sqrt{\chi(p)}\right)^n U_n\left(\frac{-a_p}{2p^{(k-1)/2}\sqrt{\chi(p)}}\right). \tag{22}$$

Aside: Some more *Mathematica* I

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Recall: $a_{p^{n+1}} = a_{p^n} a_p - \chi(p) p^{k-1} a_{p^{n-1}}$ and $a_1 = 1$.

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Output: $\ldots, a_{p^7} \to a_p a_{p^6} - Z a_{p^5}, \ldots$

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(2) Known results about Chebyshev polynomials of the second kind can now be used to derive various identities for terms in the sequence $\{a_{p^n}\}$, where p is a prime.

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Properties of Chebyshev polynomials of the second kind

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Properties of Chebyshev polynomials of the second kind I

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The following properties of the Chebyshev polynomials of the second kind are known.

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The generating function

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1-2tx+t^2}. \tag{24}$$

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The closed form

$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}}. \tag{25}$$

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$$U_n(x) = \frac{\left(x + \sqrt{x^2 - 1}\right)^{n+1} - \left(x - \sqrt{x^2 - 1}\right)^{n+1}}{2\sqrt{x^2 - 1}}. \tag{25}$$

$$U_n(x)^2 - U_{n+1}(x) U_{n-1}(x) = 1. \tag{26}$$

Properties of Chebyshev polynomials of the second kind II

Properties of Chebyshev polynomials of the second kind II

$$U_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n-j}{j} (2x)^{n-2j}. \tag{27}$$

Properties of Chebyshev polynomials of the second kind II

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For all integers $m \geq 1$ and $n \geq 0$,

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$$\sum_{n=0}^{\infty} U_n(x) \frac{t^n}{n!} = e^{tx} \left(\frac{x \sin\left(t\sqrt{1-x^2}\right)}{\sqrt{1-x^2}} + \cos\left(t\sqrt{1-x^2}\right) \right). \tag{34}$$

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$$\sum_{n=0}^{\infty} U_n(x) \frac{t^{n+1}}{(n+1)!} = e^{tx} \frac{\sin\left(t\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}, \tag{35}$$

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$$\sum_{n=0}^{\infty} U_n^2(x) t^n = \frac{(t+1)}{(1-t)\left((t+1)^2 - 4tx^2\right)}. \tag{36}$$

Define

$$F_{\pm} = xy \pm \sqrt{\left(1-x^2\right)\left(1-y^2\right)},$$
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$$\sum_{n=0}^{\infty} U_n(x) U_n(y) \frac{t^{n+1}}{(n+1)!} = \frac{e^{tF_+}\cos(t\Phi_-) - e^{tF_-}\cos(t\Phi_+)}{2\sqrt{1-x^2}\sqrt{1-y^2}}. \tag{37}$$

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$$\sum_{n=0}^{\infty} U_n(x) U_n(y) t^n = \frac{1-t^2}{\left(1-t^2\right)^2 - 4t(y-tx)(x-ty)}. \tag{38}$$

Applications to the Fourier Coefficients of Hecke Eigenforms

Application of identities for Chebyshev polynomials of the second kind I

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Let $f(q) = q + \sum_{n=2}^{\infty} a_n q^n$ be a normalized Hecke eigenform of weight k, level N, and Nebentypus χ.

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The identities in the previous section are used in conjunction with the identity

$$a_{p^n} = \left(-p^{(k-1)/2}\sqrt{\chi(p)}\right)^n U_n\left(\frac{-a_p}{2p^{(k-1)/2}\sqrt{\chi(p)}}\right), \qquad (39)$$

to derive identities for the members of the sequence $\{a_{p^n}\}$.

Formal Derivation of the Product Form of the L-Function

From

$$\sum_{n=0}^{\infty} U_n(x) t^n = \frac{1}{1 - 2tx + t^2}, \tag{40}$$

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From this, the multiplicative property, $a_m a_n = a_{mn}$ when $\gcd(m, n) = 1$, gives that

$$L(f, s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \sum_{n=0}^{\infty} \frac{a_{p^n}}{p^{sn}} = \prod_p \frac{1}{1 - a_p p^{-s} + \chi(p) p^{-2s} p^{k-1}}. \tag{42}$$

An L-function for the sequence $\{a_n^2\}$

From

$$\sum_{n=0}^{\infty} U_n^2(x) t^n = \frac{(t+1)}{(1-t)\left((t+1)^2 - 4tx^2\right)} \tag{43}$$

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$$\sum_{n=0}^{\infty} \frac{a_{p^n}^2}{p^{sn}} = \frac{1+\chi(p)p^{k-s-1}}{(1-\chi(p)p^{k-s-1})\left((1+\chi(p)p^{k-s-1})^2 - a_p^2 p^{-s}\right)}.$$

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$$L_2(f,s) := \sum_{n=1}^{\infty} \frac{a_n^2}{n^s} = \prod_p \frac{1+\chi(p)p^{k-s-1}}{(1-\chi(p)p^{k-s-1})\left((1+\chi(p)p^{k-s-1})^2 - a_p^2 p^{-s}\right)}.$$

For convergence we may take $Re(s) > k$.

Ramanujan τ-function, Example I

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Example

For any prime p and any complex s with $Re(s) > 12$,

$$\sum_{n=0}^{\infty} \frac{\tau^2(p^n)}{p^{sn}} = \frac{1+p^{11-s}}{(1-p^{11-s})\left((1+p^{11-s})^2 - \tau^2(p)p^{-s}\right)}. \qquad (44)$$

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$$\sum_{n=0}^{\infty} \frac{a_{p^n} t^n}{n!} = \exp\left(\frac{a_p t}{2}\right) \left(\cos\left(\frac{1}{2} t \sqrt{4p^{k-1}\chi(p) - a_p^2}\right) + \frac{a_p \sin\left(\frac{1}{2} t \sqrt{4p^{k-1}\chi(p) - a_p^2}\right)}{\sqrt{4p^{k-1}\chi(p) - a_p^2}} \right), \quad (45)$$

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For any prime p and any $t \in \mathbb{C}$,

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Identities from the Bivariate Generating Functions I

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$$F_{\pm} = a_{p_1} a_{p_2} \pm \sqrt{4p_1^{k-1}\chi(p_1) - a_{p_1}^2}\sqrt{4p_2^{k-1}\chi(p_2) - a_{p_2}^2},$$

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Identities from the Bivariate Generating Functions II

Theorem (continued)

For any $t \in \mathbb{C}$ *satisfying* $|t| < (p_1 p_2)^{-k/2}$,

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Ramanujan τ-function, Example III

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Example

Let p_1 and p_2 be primes (distinct or otherwise) and define

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Then for any $t \in \mathbb{C}$,

$$\sum_{n=0}^{\infty} \tau(p_1^n)\tau(p_2^n)\frac{t^{n+1}}{(n+1)!} = 2\frac{e^{t/4F_+}\cos(t/4\Phi_-) - e^{t/4F_-}\cos(t/4\Phi_+)}{\sqrt{4p_1^{11} - \tau^2(p_1)}\sqrt{4p_2^{11} - \tau^2(p_2)}}. \tag{49}$$

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$$\sum_{n=0}^{\infty} \tau(p_1^n)\tau(p_2^n)t^n$$

$$= \frac{1 - p_1^{11}p_2^{11}t^2}{\left(1 - p_1^{11}p_2^{11}t^2\right)^2 - t\left(\tau\left(p_1\right) - p_1^{11}\tau\left(p_2\right)t\right)\left(\tau\left(p_2\right) - p_2^{11}\tau\left(p_1\right)t\right)}.$$

An Identity Implying a Divisibility Property of the Sequence a_{p^n}

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Let the sequence a_{p^n} be as defined in Proposition 8.1. If $m \geq 1$ and $n \geq 2$ are integers, then

$$a_{p^{mn-1}} = a_{p^{n-1}} \times \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} (-1)^j \binom{m-1-j}{j} \left(a_{p^n} - p^{k-1}\chi(p) a_{p^{n-2}} \right)^{m-1-2j} p^{(k-1)nj} \chi^j(p). \tag{50}$$

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Remark: Note that if the numbers a_{p^n} are integers, then (50) implies that if $n+1 | m+1$, then $a_{p^n} | a_{p^m}$.

Ramanujan τ-function, Example IV

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Example

If $m \geq 1$ and $n \geq 2$ are integers, then

$$\tau(p^{mn-1}) = \tau(p^{n-1}) \times \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} (-1)^j \binom{m-1-j}{j} \left(\tau(p^n) - p^{11}\tau(p^{n-2})\right)^{m-1-2j} p^{11nj}.$$

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If $m \geq 1$ and $n \geq 2$ are integers, then

$$\tau(p^{mn-1}) = \tau(p^{n-1}) \times \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} (-1)^j \binom{m-1-j}{j} \left(\tau(p^n) - p^{11}\tau(p^{n-2})\right)^{m-1-2j} p^{11nj}.$$

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For example, taking $m = 119$ and considering the divisors of 120, then for any prime p,

$$\tau(p^n) | \tau(p^{119}) \text{ for any } n \in \{1, 2, 3, 4, 5, 7, 9, 11, 14, 19, 23, 29, 39, 59\}.$$

Some Remarks on the Speed of Convergence of some of the Series

Recall:

Example

Let p_1 and p_2 be primes (distinct or otherwise) and define

$$F_{\pm} = \tau(p_1)\tau(p_2) \pm \sqrt{4p_1^{11} - \tau^2(p_1)}\sqrt{4p_2^{11} - \tau^2(p_2)},$$
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Then for any $t \in \mathbb{C}$,

$$\sum_{n=0}^{\infty} \tau(p_1^n)\tau(p_2^n)\frac{t^{n+1}}{(n+1)!} = 2\frac{e^{t/4F_+}\cos(t/4\Phi_-) - e^{t/4F_-}\cos(t/4\Phi_+)}{\sqrt{4p_1^{11} - \tau^2(p_1)}\sqrt{4p_2^{11} - \tau^2(p_2)}}. \tag{51}$$

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$$R(2,3,1/10) = \frac{1}{72\sqrt{236929}}\Bigg[e^{\frac{1}{40}\left(144\sqrt{236929}-6048\right)} \cos\left(\frac{1}{40}\left(-2016\sqrt{119} - 432\sqrt{1991}\right)\right) - e^{\frac{1}{40}\left(-6048-144\sqrt{236929}\right)} \cos\left(\frac{1}{40}\left(2016\sqrt{119} - 432\sqrt{1991}\right)\right)\Bigg] \approx 1.977000812890026 \times 10^{690}. \quad (52)$$

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One might wonder how quickly the series converges to such a large number.

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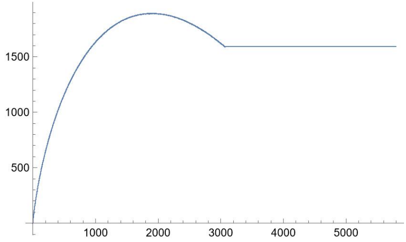

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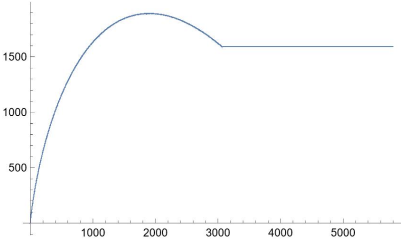

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However, the picture of convergence, which appears to show S_N getting close to the limiting value $R(2,3,1/10)$ once N gets a little above 3000, is somewhat deceptive,

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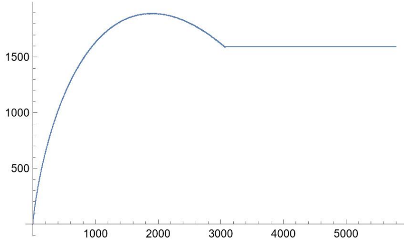

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However, the picture of convergence, which appears to show S_N getting close to the limiting value $R(2, 3, 1/10)$ once N gets a little above 3000, is somewhat deceptive, due to the fact that $R(2, 3, 1/10)$ is so large.

Speed of Convergence IV

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The following table shows the value of $S_N - R(2, 3, 1/10)$ for some values of $N \geq 2700$.

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The following table shows the value of $S_N - R(2,3,1/10)$ for some values of $N \geq 2700$.

Table 2: The convergence of S_n to $R(2,3,1/10)$

N	$S_N - R(2,3,1/10)$	N	$S_N - R(2,3,1/10)$
2700	-2.06017×10^{756}	4300	-2.76543×10^{339}
2900	2.76208×10^{723}	4500	-4.67955×10^{266}
3100	8.84248×10^{683}	4700	2.09614×10^{190}
3300	-2.04249×10^{638}	4900	1.28799×10^{110}
3500	-3.39702×10^{588}	5100	-1.13270×10^{26}
3700	-6.32613×10^{532}	5300	-1.81362×10^{-61}
3900	8.00337×10^{472}	5500	$-1.83820 \times 10^{-152}$
4100	2.20553×10^{408}	5700	8.45999×10^{-246}

References

- Dattoli, G.; Sacchetti, D.; Cesarano, C. *A note on Chebyshev polynomials.* Ann. Univ. Ferrara Sez. VII (N.S.) **47** (2001), 107–115.
- Jeffrey, A.; Dai, H.-H. *Handbook of mathematical formulas and integrals*. Fourth edition. With 1 CD-ROM (Windows and Macintosh). Elsevier/Academic Press, Amsterdam, 2008. xlvi+541 pp.
- Mason, J. C.; Handscomb, D. C. *Chebyshev polynomials.* Chapman & Hall/CRC, Boca Raton, FL, 2003. xiv+341 pp. ISBN: 0-8493-0355-9
- Rivlin, T. J. *Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory: Second Edition*, Dover Publications, Inc., Mineola, NY, 2020, viii + 249 pp.

Thanks

Thank you for listening/watching.