THE CONVERGENCE AND DIVERGENCE OF q-CONTINUED FRACTIONS OUTSIDE THE UNIT CIRCLE

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Abstract. We consider two classes of q-continued fraction whose odd and even parts are limit 1-periodic for $|q| > 1$, and give theorems which guarantee the convergence of the continued fraction, or of its odd- and even parts, at points outside the unit circle.

1. Introduction

Studying the convergence behaviour of the odd and even parts of continued fractions is interesting for a number of different reasons (see, for example, *Section 9.4* of [6]). In this present paper, we examine the convergence behaviour of q-continued fractions outside the unit circle.

Many well-known q-continued fractions have the property that their odd and even parts converge everywhere outside the unit circle. These include the Rogers-Ramanujan continued fraction,

$$K(q) := 1 + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \frac{q^4}{1} + \cdots$$

and the three Ramanujan-Selberg continued fractions studied by Zhang in [8], namely,

$$S_1(q) := 1 + \frac{q}{1} + \frac{q+q^2}{1} + \frac{q^3}{1} + \frac{q^2+q^4}{1} + \cdots,$$

$$S_2(q) := 1 + \frac{q+q^2}{1} + \frac{q^4}{1} + \frac{q^3+q^6}{1} + \frac{q^8}{1} + \cdots,$$

and

$$S_3(q) := 1 + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \frac{q^4+q^8}{1} + \cdots.$$

It was proved in [1] that if $0 < |x| < 1$ then the odd approximants of $1/K(1/x)$ tend to

$$1 - \frac{x}{1} + \frac{x^2}{1} - \frac{x^3}{1} + \cdots$$

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while the even approximants tend to

$$\frac{x}{1}+\frac{x^4}{1}+\frac{x^8}{1}+\frac{x^{12}}{1}+\cdots.$$

This result was first stated, without proof, by Ramanujan. In [8], Zhang expressed the odd and even parts of each of $S_1(q)$, $S_2(q)$ and $S_3(q)$ as infinite products, for q outside the unit circle.

Other q-continued fractions have the property that they converge everywhere outside the unit circle. The most famous example of this latter type is Göllnitz-Gordon continued fraction,

$$GG(q) := 1+q+\frac{q^2}{1+q^3}+\frac{q^4}{1+q^5}+\frac{q^6}{1+q^7}+\cdots.$$

In this present paper we study the convergence behaviour outside the unit circle of two families of q-continued fractions, families which include all of the above continued fractions.

2. Convergence of the odd and even parts of q-continued fractions outside the unit circle

Before coming to our theorems, we need some notation and some results on limit 1-periodic continued fractions.

Let the n-th approximant of the continued fraction $b_0 + K_{n=1}^{\infty} a_n/b_n$ be P_n/Q_n. The *even* part of $b_0 + K_{n=1}^{\infty} a_n/b_n$ is the continued fraction whose n-th numerator (denominator) convergent equals P_{2n} (Q_{2n}), for $n \geq 0$. The *odd* part of $b_0 + K_{n=1}^{\infty} a_n/b_n$ is the continued fraction whose zero-th numerator convergent is P_1/Q_1, whose zero-th denominator convergent is 1, and whose n-th numerator (respectively denominator) convergent equals P_{2n+1} (respectively Q_{2n+1}), for $n \geq 1$.

For later use we give explicit expressions for the odd- and even parts of a continued fraction. From [7], page 83, the even part of $b_0 + K_{n=1}^{\infty} a_n/b_n$ is given by

$$b_0+\frac{b_2a_1}{b_2b_1+a_2}-\frac{a_2a_3b_4/b_2}{a_4+b_3b_4+a_3b_4/b_2}-\frac{a_4a_5b_6/b_4}{a_6+b_5b_6+a_5b_6/b_4}-\cdots. \tag{2.1}$$

From [7], page 85, the odd part of $b_0 + K_{n=1}^{\infty} a_n/b_n$ is given by

$$\begin{aligned}\frac{b_0b_1+a_1}{b_1}&-\frac{a_1a_2b_3/b_1}{b_1(a_3+b_2b_3)+a_2b_3}-\frac{a_3a_4b_5b_1/b_3}{a_5+b_4b_5+a_4b_5/b_3}\\&-\frac{a_5a_6b_7/b_5}{a_7+b_6b_7+a_6b_7/b_5}-\frac{a_7a_8b_9/b_7}{a_9+b_8b_9+a_8b_9/b_7}-\cdots.\end{aligned} \tag{2.2}$$

Definition:. Let $t(w) = c/(1+w)$, where $c \neq 0$. Let x and y denote the fixed points of the linear fractional transformation $t(w)$. Then $t(w)$ is called

$$\begin{aligned} &(i) \text{ parabolic}, && \text{if } x = y, \\ &(ii) \text{ elliptic}, && \text{if } x \neq y \text{ and } |1+x| = |1+y|, \\ &(iii) \text{ loxodromic}, && \text{if } x \neq y \text{ and } |1+x| \neq |1+y|. \end{aligned} \tag{2.3}$$

In case (iii), if $|1+x| > |1+y|$, then $\lim_{n\to\infty} t^n(w) = x$ for all $w \neq y$, x is called the *attractive* fixed point of $t(w)$ and y is called the *repulsive* fixed point of $t(w)$.

Remark: The above definitions are usually given for more general linear fractional transformations but we do not need this full generality here.

The fixed points of $t(w) = c/(1+w)$ are $x = (-1+\sqrt{1+4c})/2$ and $y = (-1+\sqrt{1+4c})/2$. It is easy to see that $t(w)$ is parabolic only in the case $c = -1/4$, that it is elliptic only when c is a real number in the interval $(-\infty, -1/4)$ and that it is loxodromic for all other values of c.

Let $\hat{\mathbb{C}}$ denote the extended complex plane. From [7], pp. 150–151, one has the following theorem.

Theorem 1. *Suppose* $1 + K_{n=1}^{\infty} a_n/1$ *is limit* 1*-periodic, with* $\lim_{n\to\infty} a_n = c \neq 0$. *If* $t(w) = c/(1+w)$ *is loxodromic, then* $1 + K_{n=1}^{\infty} a_n/1$ *converges to a value* $f \in \hat{\mathbb{C}}$.

Remark: In the cases where $t(w)$ is parabolic or elliptic, whether $1 + K_{n=1}^{\infty} a_n/1$ converges or diverges depends on how the a_n converge to c.

We also make use of Worpitzky's Theorem (see [7], pp. 35–36).

Theorem 2. *(Worpitzky) Let the continued fraction* $K_{n=1}^{\infty} a_n/1$ *be such that* $|a_n| \leq 1/4$ *for* $n \geq 1$. *Then* $K_{n=1}^{\infty} a_n/1$ *converges. All approximants of the continued fraction lie in the disc* $|w| < 1/2$ *and the value of the continued fraction is in the disk* $|w| \leq 1/2$.

We first consider continued fractions of the form

$$\begin{aligned} G(q) : &= 1 + K_{n=1}^{\infty} \frac{a_n(q)}{1} := 1 + \frac{f_1(q^0)}{1} + \cdots + \frac{f_k(q^0)}{1} \\ &+ \frac{f_1(q^1)}{1} + \cdots + \frac{f_k(q^1)}{1} + \cdots + \frac{f_1(q^n)}{1} + \cdots + \frac{f_k(q^n)}{1} + \cdots, \end{aligned}$$

where $f_s(x) \in \mathbb{Z}[q][x]$, for $1 \leq s \leq k$. Thus, for $n \geq 0$ and $1 \leq s \leq k$,

$$a_{nk+s}(q) = f_s(q^n). \tag{2.4}$$

Many well-known q-continued fractions, including the Rogers-Ramanujan continued fraction and the three Ramanujan-Selberg continued fractions are of this form, with k at most 2. Following the example of these four continued fractions, we make the additional assumptions that, for $i \geq 1$,

$$\text{degree}(a_{i+1}(q)) = \text{degree}(a_i(q)) + m, \tag{2.5}$$

where m is a fixed positive integer, and that all of the polynomials $a_n(q)$ have the same leading coefficient. We prove the following theorem.

Theorem 3. *Suppose $G(q) = 1 + K_{n=1}^{\infty} a_n(q)/1$ is such that the $a_n := a_n(q)$ satisfy* (2.4) *and* (2.5)*. Suppose further that each $a_n(q)$ has the same leading coefficient. If $|q| > 1$ then the odd and even parts of $G(q)$ both converge.*

Remark: Worpitzky's Theorem gives only that odd- and even parts of $G(q)$ converge for those q satisfying $|(1 + q^m)(1 + q^{-m})| > 4$, a clearly weaker result.

Proof. Let $|q| > 1$. For ease of notation we write a_n for $a_n(q)$. By (2.1), the even part of $G(q)$ is given by

$$\begin{aligned}
G_e(q) &:= 1 + \frac{a_1}{1 + a_2} - \frac{a_2 a_3}{a_4 + a_3 + 1} - \frac{a_4 a_5}{a_6 + a_5 + 1} - \cdots \\
&\approx 1 + \frac{\frac{a_1}{1+a_2}}{1} - \frac{\frac{a_2 a_3}{(1+a_2)(a_4+a_3+1)}}{1} - \frac{\frac{a_4 a_5}{(a_4+a_3+1)(a_6+a_5+1)}}{1} - \cdots \\
&= 1 + K_{n=1}^{\infty} \frac{c_n}{1},
\end{aligned}$$

where, for $n \geq 3$,

$$c_n = \frac{a_{2n-2} a_{2n-1}}{(a_{2n-2} + a_{2n-3} + 1)(a_{2n} + a_{2n-1} + 1)}.$$

By (2.5), the fact that each of the $a_i(q)$'s has the same leading coefficient and the fact that if $|q| > 1$ then $\lim_{i\to\infty} 1/a_i = 0$, it follows that

$$\begin{aligned}
\lim_{n\to\infty} c_n &= \lim_{n\to\infty} \frac{1}{(1 + a_{2n-3}/a_{2n-2} + 1/a_{2n-2})(a_{2n}/a_{2n-1} + 1 + 1/a_{2n-1})} \\
&= \frac{1}{(1 + q^m)(1 + q^{-m})} := c.
\end{aligned}$$

Hence $G_e(q)$ is limit 1-periodic. Note that the value of c depends on q.

Let the fixed points of $t(w) = c/(1 + w)$ be denoted x and y. From the remarks following (2.3), it is clear that $t(w)$ is parabolic only in the case $-1/((1 + q^m)(1 + q^{-m})) = -1/4$. The only solution to this equation is $q^m = 1$, so that $t(w)$ is not parabolic for any point outside the unit circle.

Similarly, $t(w)$ is elliptic only when $-1/((1 + q^m)(1 + q^{-m}) = -1/4 - v$, for some real positive number v. The solutions to this equation satisfy $q^m = (i + \sqrt{v})/(i - \sqrt{v})$ or $q^m = (i - \sqrt{v})/(i + \sqrt{v})$. However, it is easily seen that these are points on the unit circle.

In all other cases $t(w)$ is loxodromic and $G_e(q)$ converges in $\hat{\mathbb{C}}$. This proves the result for $G_e(q)$.

Similarly, by (2.2), the odd part of $G(q)$ is given by

$$G_o(q) := \frac{1 + a_1}{1} - \frac{a_1 a_2}{a_3 + a_2 + 1} - \frac{a_3 a_4}{a_5 + a_4 + 1} - \frac{a_5 a_6}{a_7 + a_6 + 1} - \cdots.$$

The proof in this case is virtually identical. □

As an application of the above theorem, we have the following example.

Example 1. *If $|q| > 1$, then the odd and even parts of*

$$G(q) = 1 + \frac{6q}{1} + \frac{3q^2+7q}{1} + \frac{3q^3+5q^2}{1} + \frac{q^4+7q^3+3q+2}{1} + \frac{q^5+3q^4+2q^3}{1} + \frac{q^6+2q^5+7q^3}{1} + \frac{q^7+7q^5}{1} + \frac{q^8+7q^6+3q^3+2q}{1} + \cdots$$
$$\cdots + \frac{q^{4n+1}+3q^{3n+1}+2q^{2n+1}}{1} + \frac{q^{4n+2}+2q^{3n+2}+7q^{2n+1}}{1} + \frac{q^{4n+3}+5q^{3n+2}+2q^{2n+3}}{1} + \frac{q^{4n+4}+7q^{3n+3}+3q^{2n+1}+2q^n}{1} + \cdots$$

converge.

Proof. Let $k = 4$ and

$$f_1(x) = qx^4 + 3qx^3 + 2qx^2,$$
$$f_2(x) = q^2x^4 + 2q^2x^3 + 7qx^2,$$
$$f_3(x) = q^3x^4 + 5q^2x^3 + 2q^3x^2,$$
$$f_4(x) = q^4x^4 + 7q^3x^3 + 3qx^2 + 2x.$$

Then, for $n \geq 0$ and $1 \leq j \leq 4$,

$$a_{4n+j}(q) = f_j(q^n).$$

Thus (2.4) is satisfied. It is clear that (2.5) is satisfied with $M = 1$ and each $a_n(q)$ has the same leading coefficient, namely, 1. □

Remark: It is clear form Theorem 3 that if $k = 1$ and $f_i(x)$ is *any* polynomial with coefficients in $\mathbb{Z}[q]$, then the odd and even parts of $1 + K_{n=0}^{\infty} f_1(q^n)/1$ converge everywhere outside the unit circle to values in $\hat{\mathbb{C}}$, since all the conditions of the theorem are satisfied automatically, at least for a tail of the continued fraction.

We also consider continued fractions of the form

$$G(q) := b_0(q) + K_{n=1}^{\infty} \frac{a_n(q)}{b_n(q)}$$
$$:= g_0(q^0) + \frac{f_1(q^0)}{g_1(q^0)} + \cdots + \frac{f_{k-1}(q^0)}{g_{k-1}(q^0)} + \frac{f_k(q^0)}{g_0(q^1)}$$
$$+ \frac{f_1(q^1)}{g_1(q^1)} + \cdots + \frac{f_{k-1}(q^1)}{g_{k-1}(q^1)} + \frac{f_k(q^1)}{g_0(q^2)} +$$
$$\cdots + \frac{f_k(q^{n-1})}{g_0(q^n)} + \frac{f_1(q^n)}{g_1(q^n)} + \cdots + \frac{f_{k-1}(q^n)}{g_{k-1}(q^n)} + \frac{f_k(q^n)}{g_0(q^{n+1})} + \cdots$$

where $f_s(x), g_{s-1}(x) \in \mathbb{Z}[q][x]$, for $1 \leq s \leq k$. Thus, for $n \geq 0$ and $1 \leq s \leq k$,

$$a_{nk+s}(q) = f_s(q^n), \qquad b_{nk+s-1}(q) = g_{s-1}(q^n). \tag{2.6}$$

An example of a continued fraction of this type is the Göllnitz-Gordon continued fraction (with $k = 1$).

We suppose that degree $(a_1(q)) = r_1$, degree $(b_0(q)) = r_2$, and that, for $i \geq 1$,

$$\begin{aligned} \text{degree}(a_{i+1}(q)) &= \text{degree}(a_i(q)) + a, \\ \text{degree}(b_i(q)) &= \text{degree}(b_{i-1}(q)) + b, \end{aligned} \tag{2.7}$$

where a and b are fixed positive integers and r_1 and r_2 are non-negative integers. Condition 2.7 means that, for $n \geq 1$,

$$\text{degree}(a_n(q)) = (n-1)a + r_1, \qquad \text{degree}(b_n(q)) = n\,b + r_2. \tag{2.8}$$

We also supposed that each $a_n(q)$ has the same leading coefficient L_a and that each $b_n(q)$ has the same leading coefficient L_b.

For such continued fractions we have the following theorem.

Theorem 4. *Suppose $G(q) = b_o + K_{n=1}^{\infty} a_n(q)/b_n(q)$ is such that the $a_n := a_n(q)$ and the $b_n := b_n(q)$ satisfy (2.6) and (2.7). Suppose further that each $a_n(q)$ has the same leading coefficient L_a and that each $b_n(q)$ has the same leading coefficient L_b. If $2b > a$ then $G(q)$ converges everywhere outside the unit circle. If $2b = a$, then $G(q)$ converges outside the unit circle to values in $\hat{\mathbb{C}}$, except possibly at points q satisfying $L_b^2/L_a q^{b-r_1+2r_2} \in [-4, 0)$. If $2b < a$, then the odd and even parts of $G(q)$ converge everywhere outside the unit circle.*

Proof. . Let $|q| > 1$. We first consider the case $2b > a$. By a simple transformation, we have that

$$b_0 + K_{n=1}^{\infty} \frac{a_n}{b_n} \approx b_0 + \frac{a_1/b_1}{1} {}_{+} K_{n=2}^{\infty} \frac{a_n/(b_n b_{n-1})}{1}.$$

Since $2b > a$, $a_n/(b_n b_{n-1}) \to 0$ as $n \to \infty$, and $G(q)$ converges to a value in $\hat{\mathbb{C}}$, by Worpitzky's theorem.

Suppose $2b = a$. Then, by (2.7), (2.8) and the fact that each $a_n(q)$ has the same leading coefficient L_a and that each $b_n(q)$ has the same leading coefficient L_b,

$$\lim_{n\to\infty} \frac{a_n}{b_n b_{n-1}} = \frac{L_a}{L_b^2 q^{b-r_1+2\,r_2}} := c.$$

Note once again that the value of c depends on q. Once again, by the remarks following (2.3), the linear fractional transformation $t(w) = c/(1 + w)$ is parabolic only in the case $L_a/(L_b^2 q^{b-r_1+2\,r_2}) = -1/4$ or $q^{b-r_1+2r_2} = -4L_a/L_b^2$.

Similarly, $t(w)$ is elliptic only when $q^{-b+r_1-2r_2}\,L_a/{L_b}^2 \in (-\infty,-1/4)$, or

$$q^{b-r_1+2r_2} = \frac{-4\,L_a}{(1+4\,v)\;{L_b}^2},$$

for some real positive number v. In other words, $t(w)$ is elliptic (for $|q|>1$) only when $q^{b-r_1+2r_2}$ lies either in the open interval $(-4L_a/L_b^2, 0)$ or $(0,-4L_a/L_b^2)$, depending on the sign of L_a. In all other cases, $t(w)$ is loxodromic, and $G(q)$ converges.

Suppose $2\,b<a$. From (2.1) it is clear that the even part of $G(q) = b_0 + K_{n=1}^{\infty} a_n/b_n$ can be transformed into the form $b_0 + K_{n=1}^{\infty} c_n/1$, where, for $n \geq 3$,

$$\begin{aligned} c_n &= \frac{-a_{2n-2}a_{2n-1}\dfrac{b_{2n}}{b_{2n-2}}}{\left(a_{2n-2}+b_{2n-3}b_{2n-2}+a_{2n-3}\dfrac{b_{2n-2}}{b_{2n-4}}\right)\left(a_{2n}+b_{2n-1}b_{2n}+a_{2n-1}\dfrac{b_{2n}}{b_{2n-2}}\right)} \\ &= \frac{\dfrac{-a_{2n-1}b_{2n}}{a_{2n}b_{2n-2}}}{\left(1+\dfrac{b_{2n-3}b_{2n-2}}{a_{2n-2}}+\dfrac{a_{2n-3}b_{2n-2}}{a_{2n-2}b_{2n-4}}\right)\left(1+\dfrac{b_{2n-1}b_{2n}}{a_{2n}}+\dfrac{a_{2n-1}b_{2n}}{a_{2n}b_{2n-2}}\right)}. \end{aligned}$$

Once again using (2.7), (2.8) and the fact that each $a_n(q)$ has the same leading coefficient L_a and that each $b_n(q)$ has the same leading coefficient L_b, we have that

$$\lim_{n\to\infty} c_n = -\frac{q^{2b-a}}{(1+q^{2b-a})^2} := c.$$

The linear fractional transformation $t(w) = c/(1+w)$ is parabolic only in the case $-q^{2b-a}/(1+q^{2b-a})^2 = -1/4$ or $q^{2b-a} = 1$, and thus $|q| = 1$. It is elliptic only when $-q^{2b-a}/(1+q^{2b-a})^2 \in (-\infty,-1/4)$, and a simple argument shows that this implies that $|q^{2b-a}| = 1$, and again $|q| = 1$.

In all other cases $t(w)$ is loxodromic, and the even part of $G(q)$ converges by Theorem 1.

The proof for the odd part of $G(q)$ is very similar and is omitted. □

Remarks: (1) Worpitzky's Theorem once again gives weaker results. In the example below, for example, Worpitzky's Theorem gives that $G(q)$ converges for $|q|>4$, in contrast to the result from our theorem, which says that $G(q)$ converges everywhere outside the unit circle, except possibly for $q \in [-4,-1)$.

(2) In some cases the result is the best possible. Numerical evidence suggests that the continued fraction below converges nowhere in the interval $(-4,-1)$.

As an application of Theorem 4, we have the following example.

Example 2. *If* $|q|>1$, *then*

$$G(q) = q + 2 + \frac{6q^2}{q^2+2} + \frac{3q^4+7q^2}{q^3+2} + \frac{3q^6+5q^4}{q^4+2} + \frac{q^8+7q^6+3q^2+2}{q^5+q+1} + \frac{q^{10}+3q^8+2q^6}{q^6+q^2+1} + \frac{q^{12}+2q^{10}+7q^6}{q^7+q^2+1} + \frac{q^{14}+7q^{10}}{q^8+q^3} + \frac{q^{16}+7q^{12}+3q^6+2q^2}{q^9+q^2+1} + \cdots + \frac{q^{8n+2}+3q^{6n+2}+2q^{4n+2}}{q^{4n+2}+q^{2n}+1} + \frac{q^{8n+4}+2q^{6n+4}+7q^{4n+2}}{q^{4n+3}+q^{2n}+1} + \frac{q^{8n+6}+5q^{6n+4}+2q^{4n+6}}{q^{4n+4}+q^{3n}+1} + \frac{q^{8n+8}+7q^{6n+6}+3q^{4n+2}+2q^{2n}}{q^{4(n+1)+1}+q^{n+1}+1} + \cdots$$

converges, except possibly for $q \in [-4,-1)$.

Proof. Let $k = 4$ and

$$\begin{aligned} f_1(x) &= q^2x^8 + 3q^2x^6 + 2q^2x^4, \\ f_2(x) &= q^4x^8 + 2q^4x^6 + 7q^2x^4, \\ f_3(x) &= q^6x^8 + 5q^4x^6 + 2q^6x^4, \\ f_4(x) &= q^8x^8 + 7q^6x^6 + 3q^2x^4 + x^2 \\ g_0(x) &= qx^4 + x + 1, \\ g_1(x) &= q^2x^4 + x^2 + 1, \\ g_2(x) &= q^3x^4 + x^2 + 1, \\ g_3(x) &= q^4x^4 + x^3 + 1. \end{aligned}$$

Then, for $n \geq 0$ and $1 \leq j \leq 4$,

$$\begin{aligned} a_{4n+j}(q) &= f_j(q^n), \\ b_{4n+j-1}(q) &= g_{j-1}(q^n). \end{aligned}$$

The other requirements of the theorem are satisfied, with $L_a = L_b = 1$, $a = 2$, $b = 1$, $r_1 = 2$ and $r_2 = 1$. Therefore $b - r_1 + 2r_2 = 1$, $L_a/L_b^2 = 1$ and $G(q)$ converges outside the unit circle, except possibly for $q \in [-4,-1)$. □

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