

ON THE DIVERGENCE IN THE GENERAL SENSE OF q -CONTINUED FRACTION ON THE UNIT CIRCLE.

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ABSTRACT. We show, for each q -continued fraction $G(q)$ in a certain class of continued fractions, that there is an uncountable set of points on the unit circle at which $G(q)$ diverges in the general sense. This class includes the Rogers-Ramanujan continued fraction and the three Ramanujan-Selberg continued fraction.

We discuss the implications of our theorems for the general convergence of other q -continued fractions, for example the Göllnitz-Gordon continued fraction, on the unit circle.

1. INTRODUCTION

In [2], we made a detailed study of the convergence behaviour of the famous Rogers-Ramanujan continued fraction $K(q)$, where

$$(1.1) \quad K(q) := 1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \dots}}}$$

It is an easy consequence of Worpitzky's Theorem (see [8], pp. 35–36) that $R(q)$ converges to a value in $\hat{\mathbb{C}}$ for any q inside the unit circle.

Theorem 1. (*Worpitzky*) *Let the continued fraction $K_{n=1}^{\infty} a_n/1$ be such that $|a_n| \leq 1/4$ for $n \geq 1$. Then $K_{n=1}^{\infty} a_n/1$ converges. All approximants of the continued fraction lie in the disc $|w| < 1/2$ and the value of the continued fraction is in the disc $|w| \leq 1/2$.*

Suppose $|q| > 1$. For $n \geq 1$, define

$$K_n(q) := 1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \dots + \frac{q^n}{1}}}}$$

Then

$$\lim_{j \rightarrow \infty} K_{2j+1}(q) = \frac{1}{K(-1/q)},$$
$$\lim_{j \rightarrow \infty} K_{2j}(q) = \frac{K(1/q^4)}{q}.$$

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This was stated by Ramanujan without proof and proved by Andrews, Berndt, Jacobson and Lamphere in 1992 [1].

This leaves the question of convergence on the unit circle. The convergence behaviour at roots of unity was investigated by Schur, who showed in [11] that if q is a primitive m -th root of unity, where $m \equiv 0 \pmod{5}$, then $K(q)$ diverges and if q is a primitive m -th root of unity, $m \not\equiv 0 \pmod{5}$, then $K(q)$ converges and

$$(1.2) \quad K(q) = \lambda q^{(1-\lambda\sigma m)/5} K(\lambda),$$

where $\lambda = \left(\frac{m}{5}\right)$ (the Legendre symbol) and σ is the least positive residue of $m \pmod{5}$. Note that $K(1) = \phi = (\sqrt{5} + 1)/2$, and $K(-1) = 1/\phi$.

Remark: Schur's result was essentially proved by Ramanujan, probably earlier than Schur (see [9], p.383). However, he made a calculational error (see [6], p.56).

There remains the question of whether the Rogers-Ramanujan continued fraction converges or diverges at a point on the unit circle which is not a root of unity. The chief difficulty in trying to apply the usual convergence/divergence tests stems from the facts that the Rogers-Ramanujan continued fraction converges at a set of points that is dense on the unit circle and diverges at another such dense set. This is clear from the result of Schur above.

This question about convergence on the unit circle at points which were not roots of unity remained unanswered until our paper, [2], where we showed the existence of an uncountable set of points on the unit circle at which the Rogers-Ramanujan continued fraction diverged.

To discuss this topic we use the following notation. Let the regular continued fraction expansion of any irrational $t \in (0, 1)$ be denoted by $t = [0, e_1(t), e_2(t), \dots]$. Let the i -th approximant of this continued fraction expansion be denoted by $c_i(t)/d_i(t)$. We will sometimes write e_i for $e_i(t)$, c_i for $c_i(t)$ etc, if there is no danger of ambiguity. Let $\phi = (\sqrt{5} + 1)/2$. In [2], we proved the following theorem.

Theorem 2. [2] *Let*

$$(1.3) \quad S = \{t \in (0, 1) : e_{i+1}(t) \geq \phi^{d_i(t)} \text{ infinitely often}\}.$$

Then S is an uncountable set of measure zero and, if $t \in S$ and $y = \exp(2\pi it)$, then the Rogers-Ramanujan continued fraction diverges at y .

We were also able to give explicit examples of points y on the unit circle at which $K(y)$ diverges.

Corollary 1. *Let t be the number with continued fraction expansion equal $[0, e_1, e_2, \dots]$, where e_i is the integer consisting of a tower of i twos with an i an top.*

$$t = [0, 2, 2^{2^2}, 2^{2^{2^3}}, \dots] =$$

Define the chordal metric d on $\hat{\mathbb{C}}$ by

$$(2.1) \quad d(w, z) = \frac{|z - w|}{\sqrt{1 + |w|^2} \sqrt{1 + |z|^2}}$$

when w and z are both finite, and

$$d(w, \infty) = \frac{1}{\sqrt{1 + |w|^2}}.$$

Definition: The continued fraction M is said to *converge generally* to $f \in \hat{\mathbb{C}}$ if there exist sequences $\{v_n\}, \{w_n\} \subset \hat{\mathbb{C}}$ such that $\liminf d(v_n, w_n) > 0$ and

$$\lim_{n \rightarrow \infty} S_n(v_n) = \lim_{n \rightarrow \infty} S_n(w_n) = f.$$

Remark: Jacobson shows in [7] that, if a continued fraction converges in the general sense, then the limit is unique.

The idea of general convergence is of great significance because classical convergence implies general convergence (take $v_n = 0$ and $w_n = \infty$, for all n), but the converse does not necessarily hold. General convergence is a natural extension of the concept of classical convergence for continued fractions.

We consider continued fractions of the form

$$(2.2) \quad G(q) := b_0(q) + K_{n=1}^{\infty} \frac{a_n(q)}{b_n(q)} \\ := g_0(q^0) + \frac{f_1(q^0)}{g_1(q^0)} + \cdots + \frac{f_{k-1}(q^0)}{g_{k-1}(q^0)} + \frac{f_k(q^0)}{g_0(q^1)} \\ + \frac{f_1(q^1)}{g_1(q^1)} + \cdots + \frac{f_{k-1}(q^1)}{g_{k-1}(q^1)} + \frac{f_k(q^1)}{g_0(q^2)} + \\ \cdots + \frac{f_k(q^{n-1})}{g_0(q^n)} + \frac{f_1(q^n)}{g_1(q^n)} + \cdots + \frac{f_{k-1}(q^n)}{g_{k-1}(q^n)} + \frac{f_k(q^n)}{g_0(q^{n+1})} + \cdots$$

where $f_s(x), g_{s-1}(x) \in \mathbb{Z}[q][x]$, for $1 \leq s \leq k$. Thus, for $n \geq 0$ and $1 \leq s \leq k$,

$$(2.3) \quad a_{nk+s} = a_{nk+s}(q) = f_s(q^n), \quad b_{nk+s-1} = b_{nk+s-1}(q) = g_{s-1}(q^n).$$

Many well-known q -continued fractions, including the Rogers-Ramanujan continued fraction, the three Ramanujan-Selberg continued fractions studied by Zhang in [13], and the Göllnitz-Gordon continued fraction,

$$(2.4) \quad GG(q) := 1 + q + \frac{q^2}{1 + q^3} + \frac{q^4}{1 + q^5} + \frac{q^6}{1 + q^7} + \cdots,$$

have the form of the continued fraction at (2.2), with k at most 2. It seems natural to consider a class of continued fractions which, in a sense, contains all of the above continued fractions.

For the remainder of the paper $P_n(q)/Q_n(q)$ denotes the n -th approximant of $G(q)$, P_n/Q_n if there is no danger of ambiguity. For later use, we recall some basic facts about continued fractions. It is well known (see, for

example, [8], p.9) that the P_n 's and Q_n 's satisfy the following recurrence relations.

$$(2.5) \quad \begin{aligned} P_n &= b_n P_{n-1} + a_n P_{n-2}, \\ Q_n &= b_n Q_{n-1} + a_n Q_{n-2}. \end{aligned}$$

It is also well known (see also [8], p.9) that, for $n \geq 1$,

$$(2.6) \quad P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} \prod_{i=1}^n a_i.$$

Condition 2.3 also implies that if q is a primitive m -th root of unity then $G(q)$ is periodic with period mk . Indeed, if q is a primitive m -th root of unity and $j \geq 0$,

$$(2.7) \quad a_{jmk+r} = f_r(q^{jm}) = f_r(q^0) = a_r,$$

$$b_{jmk+r} = g_r(q^{jm}) = g_r(q^0) = b_r.$$

We now assume certain facts about the approximants of $G(q)$, and the convergence behaviour of $G(q)$, at certain roots of unity.

We assume that there is a positive integer d and an integer $s \in \{1, 2, \dots, d\}$, such that if $m \equiv s \pmod{d}$ and $(r, m) = 1$, then

$$(2.8) \quad q = \exp(2\pi ir/m) \implies \begin{cases} a_n(q) \neq 0, \text{ for } n \geq 1, \\ G(q) \text{ converges and } G(q) \neq 0. \end{cases}$$

This integer s will be referred to frequently in what follows.

We further assume that if $G(q)$ converges at $q = \exp(2\pi ir/m)$, a primitive m -th root of unity, then $G(q)$ converges at any $q' = \exp(2\pi ir'/m')$, a primitive m' -th root of unity, where $m \equiv m' \pmod{d}$ and $r \equiv r' \pmod{d}$.

We also assume that there exists $\eta \in \mathbb{Q}$ such that if $H(q) := q^\eta/G(q)$ and $G(q)$ converges at $q = \exp(2\pi ir/m)$ then

$$(2.9) \quad H(\exp(2\pi ir/m)) = H(\exp(2\pi ir'/m')),$$

with r' and m' as above. Note that the above condition implies that $H(q)$ takes only finitely many values at roots of unity. Let these values be denoted H_1, H_2, \dots, H_{N_G} .

We assume that for all $m \equiv s \pmod{d}$ that there are integers K_0, K_1, K_2, K_3 and K_4 , depending only on s , such that

$$(2.10) \quad \begin{aligned} a_{km}(q) &= K_0, & P_{km-1}(q) &= K_1, \\ Q_{km-2}(q) &= K_2, & Q_{km-1}(q)P_{km-2}(q) &= K_3, \\ |Q_{km-1}(q)| &= |P_{km-2}(q)| & &= K_4 \end{aligned}$$

Here k is the positive integer in the definition of the continued fraction $G(q)$ at (2.2).

Finally, it is also assumed that there exists $r \neq u \in \{0, 1, \dots, d-1\}$, such that

$$(2.11) \quad H(\exp(2\pi ir/s)) = H_a \neq H_b = H(\exp(2\pi iu/s)),$$

for some $a, b \in \{1, \dots, N_G\}$.

It may be instructive at this point to show how these abstract conditions above apply to a particular continued fraction. We let $G(q) = K(q)$.

If we compare the continued fractions at (1.1) and (2.2), it is clear that we can take $k = 1$, $g_0(x) \equiv 1$ and $f_1(x) \equiv x$ (giving $b_n(q) = g_0(q^n) = 1$ and $a_n(q) = f_1(q^n) = q^n$).

From Schur's paper [11] (or see Table 1, which contains the relevant information from [11]) we can take $d = 5$ and $s = 1$ and if q is a primitive m -th root of unity, $m \equiv 1 \pmod{5}$, then $K(q)$ converges, giving Condition 2.8 above.

If we set $\eta = 1/5$ and set $H(q) = q^{1/5}/K(q)$, we have from (1.2) that, if q is a primitive m -th root of unity, $m \not\equiv 0 \pmod{5}$, then

$$(2.12) \quad H(q) = \frac{q^{(\lambda\sigma m)/5}}{\lambda K(\lambda)},$$

where $\lambda = \left(\frac{m}{5}\right)$ (the Legendre symbol) and σ is the least positive residue of $m \pmod{5}$. It follows that $H(q)$ can take only ten possible values at roots of unity.

It is also clear from (2.12) that Conditions 2.9 and 2.11 are satisfied, since if $m \equiv 1 \pmod{5}$ (so that $\lambda = \sigma = 1$ and $K(1) = (1 + \sqrt{5})/2$) and $q = \exp(2\pi ir/m)$, with $(r, m) = 1$, then

$$H(q) = \frac{2 \exp(2\pi ir/5)}{1 + \sqrt{5}}.$$

If q is a primitive m -th root of unity, it follows from (1.1) and (2.10) that $K_0 = a_m(q) = q^m = 1$. It follows from Schur's paper [11] (or, once again, from Table 1) that $K_1 = P_{m-1}(q) = 1$, $K_2 = Q_{m-2}(q) = 0$ and $K_3 = P_{m-2}(q)Q_{m-1}(q) = q^{(1-m)/5}q^{(m-1)/5} = 1 = K_4$. Thus Condition 2.10 is satisfied.

From the paper of Zhang [13], each of $S_1(q)$, $S_2(q)$ and $S_3(q)$ also satisfy a set of conditions of the form set out in (2.8) to (2.11). The relevant details are found in Table 1.

As before, let the regular continued fraction expansion of an irrational $t \in (0, 1)$ be denoted by $[0, e_1, e_2, \dots]$ and let the n -th approximant of this continued fraction be denoted by c_n/d_n . We prove the following theorem.

Theorem 3. *Let $G(q)$ have the form given by (2.2) and satisfy conditions (2.3) and (2.8) – (2.11).*

There exists an integer N' and a strictly increasing function $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ such that if t is any irrational in $(0, 1)$ for which there exist two subsequences of approximants $\{c_{f_n}/d_{f_n}\}$ and $\{c_{g_n}/d_{g_n}\}$ satisfying

$$(2.13) \quad \begin{aligned} c_{f_n} &\equiv r \pmod{d}, & c_{g_n} &\equiv u \pmod{d}, \\ d_{f_n} &\equiv s \pmod{d}, & d_{g_n} &\equiv s \pmod{d}. \end{aligned}$$

and

$$(2.14) \quad e_{h_{n+1}} > 2\pi\gamma(k N' d_{h_n}^2),$$

for all n , where $h_n = f_n$ or g_n . Then $H(\exp(2\pi it))$ does not converge generally.

Let S^\diamond denote the set of all $t \in (0, 1)$ satisfying (2.13) and (2.14) and set

$$(2.15) \quad Y_G = \{\exp(2\pi it) : t \in S^\diamond\}.$$

Then Y_G is an uncountable set of measure zero.

We show, as a corollary to this theorem, for each of the continued fractions $K(q)$, $S_1(q)$, $S_2(q)$ and $S_3(q)$, that there exists an uncountable set of points on the unit circle at which the continued fraction does not converge generally.

The main idea of the proof will be to show that there exist points y on the unit circle for which there exist two sequences of positive integers, $\{m_i\}$ and $\{n_i\}$, such that the subsequences of approximants to $H(y)$, $\{P_{n_i}/Q_{n_i}\}$ and $\{P_{n_i-1}/Q_{n_i-1}\}$ each tend to the same limit, L_1 say, and the subsequences $\{P_{m_i}/Q_{m_i}\}$ and $\{P_{m_i-1}/Q_{m_i-1}\}$ each tend to the same limit $L_2 \neq L_1$. This is done by constructing real numbers t in the interval $(0, 1)$ whose continued fraction expansions have a certain rapid convergence behavior and then setting $y = \exp(2\pi it)$. In addition, it is shown that the sequences $\{Q_{n_i}/Q_{n_i-1}\}$ and $\{Q_{m_i}/Q_{m_i-1}\}$ are bounded from above, for i sufficiently large. These two conditions are then shown to imply that $H(q)$ does not converge generally at y .

We first give some technical lemmas. The proofs are not given if the results are well known. Our aim is to estimate $P_i(q)$ and $Q_i(q)$ for sequences of i 's in certain arithmetic progressions. We use matrix notation since the proofs are simpler.

Lemma 1. *Let $G(q)$ be as in (2.2). There exist strictly increasing sequences of positive integers $\{\kappa_n\}$ and $\{\nu_n\}$ such that if x and y are any two points on the unit circle then, for all integers $n \geq 0$,*

$$(2.16) \quad |Q_n(x) - Q_n(y)| \leq \kappa_n |x - y|,$$

and

$$(2.17) \quad |P_n(x) - P_n(y)| \leq \nu_n |x - y|.$$

Proof. Let $\{f_n(q)\}$ be any sequence of polynomials in $\mathbb{Z}[q]$. Suppose $f_n(q) = \sum_{i=0}^{M_n} \gamma_i q^i$, where the γ_i 's are in \mathbb{Z} . Then

$$|f_n(x) - f_n(y)| \leq \sum_{i=1}^{M_n} |\gamma_i| |x^i - y^i|$$

$$\leq \sum_{i=1}^{M_n} i |\gamma_i| |x - y|.$$

Now set $\delta_n = \max \left\{ \sum_{i=1}^{M_n} i |\gamma_i|, 1, \delta_{n-1} + 1 \right\}$. Inequality (2.16) follows by setting $f_n(q) = Q_n(q)$ and $\delta_n = \kappa_n$. The result for (2.17) follows similarly. \square

With κ_n and ν_n as in the above lemma, define, for each $n \geq 1$,

$$(2.18) \quad \gamma(n) := \max\{\kappa_n, \nu_n\}.$$

This function will be used later in the proof of Theorem 3.

Lemma 2. ([12], p. 238) For $n \geq 0$,

$$(2.19) \quad \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \prod_{i=1}^n \begin{pmatrix} b_i & 1 \\ a_i & 0 \end{pmatrix}.$$

Proof. This follows, by induction, from the recurrence relations (2.5). \square

We now assume that q is a primitive m -th root of unity, $q = \exp(2\pi i n/m)$, where $(n, m) = 1$, $m \equiv s \pmod{d}$ and either $n \equiv r \pmod{d}$ and $n \equiv u \pmod{d}$, where r, s and u are as in condition (2.11).

Lemma 3. For $j \geq 1$ and $1 \leq r \leq km$,

$$(2.20) \quad \begin{pmatrix} P_{jkm+r} & P_{jkm-1+r} \\ Q_{jkm+r} & Q_{jkm-1+r} \end{pmatrix} \\ = \begin{pmatrix} P_{km-1} & a_{km} P_{km-2} \\ Q_{km-1} & a_{km} Q_{km-2} \end{pmatrix} \begin{pmatrix} P_{(j-1)km+r} & P_{(j-1)km-1+r} \\ Q_{(j-1)km+r} & Q_{(j-1)km-1+r} \end{pmatrix}.$$

For $j \geq 1$ and $0 \leq r \leq km - 1$,

$$(2.21) \quad \begin{pmatrix} P_{jkm+r} \\ Q_{jkm+r} \end{pmatrix} = \begin{pmatrix} P_{km-1} & a_{km} P_{km-2} \\ Q_{km-1} & a_{km} Q_{km-2} \end{pmatrix}^j \begin{pmatrix} P_r \\ Q_r \end{pmatrix}.$$

Proof. By Lemma 2 and the periodicity of the a_i 's and/or the b_i 's noted at (2.7), we have that

$$\begin{aligned} \begin{pmatrix} P_{jkm+r} & P_{jkm-1+r} \\ Q_{jkm+r} & Q_{jkm-1+r} \end{pmatrix} &= \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \prod_{i=1}^{jkm+r} \begin{pmatrix} b_i & 1 \\ a_i & 0 \end{pmatrix} \\ &= \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \prod_{i=1}^{km} \begin{pmatrix} b_i & 1 \\ a_i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -b_0 \end{pmatrix} \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \prod_{i=km+1}^{jkm+r} \begin{pmatrix} b_i & 1 \\ a_i & 0 \end{pmatrix} \\ &= \begin{pmatrix} P_{km} & P_{km-1} \\ Q_{km} & Q_{km-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -b_{km} \end{pmatrix} \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \prod_{i=1}^{(j-1)km+r} \begin{pmatrix} b_i & 1 \\ a_i & 0 \end{pmatrix}. \end{aligned}$$

Statement (2.20) then follows from the facts that $P_{km} = b_{km}P_{km-1} + a_{km}P_{km-2}$ and $Q_{km} = b_{km}Q_{km-1} + a_{km}Q_{km-2}$ and Lemma 2. Statement (2.21) is an immediate consequence of (2.20). \square

Remark: It is clear from (2.21), that if $G(q)$ converges then $Q_{km-1} \neq 0$, since otherwise $Q_{jkm-1} = 0$ for $j \geq 1$.

Define

$$(2.22) \quad M := \begin{pmatrix} P_{km-1} & a_{km}P_{km-2} \\ Q_{km-1} & a_{km}Q_{km-2} \end{pmatrix}.$$

Equation (2.6) implies that

$$(2.23) \quad \text{Det}(M) = a_{km}(P_{km-1}Q_{km-2} - P_{km-2}Q_{km-1}) = (-1)^{km} \prod_{i=1}^{km} a_i.$$

Let T denote the trace of M and D its determinant. In light of (2.23) and (2.10) it is clear that T and D are both integers that depend only on s . From this it is clear that

$$(2.24) \quad T^2 - 4D = K_5, \text{ for some } K_5 \in \mathbb{Z},$$

and that K_5 also depends only on s . The eigenvalues of M are

$$(2.25) \quad \begin{aligned} \lambda_1 &= \frac{T + \sqrt{T^2 - 4D}}{2}, \\ \lambda_2 &= \frac{T - \sqrt{T^2 - 4D}}{2}. \end{aligned}$$

The corresponding eigenvectors are

$$(2.26) \quad \mathbf{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{P_{km-1} - a_{km}Q_{km-2} - \sqrt{T^2 - 4D}}{2Q_{km-1}} \\ 1 \end{pmatrix}$$

and

$$(2.27) \quad \mathbf{y} := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{P_{km-1} - a_{km}Q_{km-2} + \sqrt{T^2 - 4D}}{2Q_{km-1}} \\ 1 \end{pmatrix}.$$

As shown above, if $G(q)$ converges, then $Q_{km-1} \neq 0$, and this justifies taking $x_2 = y_2 = 1$. Note for later use that $|x_1|$, $|y_1|$, λ_1 and λ_2 depend only on s . This follows from (2.10) and (2.24).

Lemma 4. *Let the eigenvalues of*

$$M = \begin{pmatrix} P_{km-1} & a_{km}P_{km-2} \\ Q_{km-1} & a_{km}Q_{km-2} \end{pmatrix}$$

be λ_1 and λ_2 . If $G(q)$ converges then $\lambda_1 = \lambda_2$ or $|\lambda_1| \neq |\lambda_2|$.

Proof. Since $a_i \neq 0$ for $i \geq 1$, it follows from (2.6), that $\text{Det}(M) \neq 0$, so that neither of the eigenvalues is zero.

Suppose $\lambda_1 \neq \lambda_2$ but $|\lambda_1| = |\lambda_2|$. In this case it is clear from (2.25) and (2.26) that \mathbf{x} and \mathbf{y} are linearly independent. For $r \in \{0, 1, 2, \dots, km-1\}$, suppose that $(P_r, Q_r)^T = p_r \mathbf{x} + q_r \mathbf{y}$, for some $p_r, q_r \in \mathbb{C}$. Then it follows from (2.21), (2.26) and (2.27) that

$$(2.28) \quad \begin{pmatrix} P_{jmk+r} \\ Q_{jmk+r} \end{pmatrix} = \begin{pmatrix} p_r \lambda_1^j x_1 + q_r \lambda_2^j y_1 \\ p_r \lambda_1^j + q_r \lambda_2^j \end{pmatrix}.$$

By some simple algebraic manipulation,

$$(2.29) \quad \frac{P_{jkm+r}}{Q_{jkm+r}} = y_1 + \frac{p_r (x_1 - y_1)}{p_r + q_r \left(\frac{\lambda_2}{\lambda_1}\right)^j}.$$

The right hand side does not converge as $j \rightarrow \infty$, unless $p_r = 0$ or $q_r = 0$, for each r .

Since we are considering the case where no $a_i = 0$, then $a_1 \neq 0$, $(P_0, Q_0) \neq \gamma(P_1, Q_1)$, for any $\gamma \in \mathbb{C}$. Hence $p_0 q_1 - p_1 q_0 \neq 0$.

We first consider the case $p_0 = 0$. Then $\lim_{j \rightarrow \infty} P_{jkm} / Q_{jkm} = y_1$. Since $p_0 = 0$, it follows from the remark above that $p_1 \neq 0$, and then it must be that $q_1 = 0$ and $\lim_{j \rightarrow \infty} P_{jkm+1} / Q_{jkm+1} = x_1 \neq y_1$, which is a contradiction.

On the other hand, if $p_0 \neq 0$, then $q_0 = 0$ and so $q_1 \neq 0$ which necessitates $p_1 = 0$, and a similar contradiction follows. This completes the proof. \square

Remarks: 1) The eigenvalues for the Rogers-Ramanujan continued fraction and the Ramanujan-Selberg continued fractions are non-zero and distinct.

2) It follows similarly from (2.29), (2.26) and (2.27), that, in the case $|\lambda_1| \neq |\lambda_2|$,

$$(2.30) \quad G(q) = \begin{cases} y_1 = \frac{P_{km-1} - a_{km}Q_{km-2} + \sqrt{T^2 - 4D}}{2Q_{km-1}}, & |\lambda_1| < |\lambda_2|, \\ x_1 = \frac{P_{km-1} - a_{km}Q_{km-2} - \sqrt{T^2 - 4D}}{2Q_{km-1}}, & |\lambda_1| > |\lambda_2|. \end{cases}$$

For later use we evaluate $G(q)$ when $\lambda_1 = \lambda_2$. In this case $T^2 - 4D = 0$. This equation implies

$$(2.31) \quad P_{km-2} = -\frac{(P_{km-1} - a_{km}Q_{km-2})^2}{4a_{km}Q_{km-1}}.$$

This in turn means that $P_{km-1} \neq a_{km}Q_{km-2}$, or else $P_{km-2} = 0$ and (2.21) gives that $P_{jkm-2} = 0$ for $j \geq 1$, implying that $G(q) = 0$, contradicting our assumption.

For ease of notation we write $P_{km-1} = a$, $Q_{km-1} = c$ and $a_{km}Q_{km-2} = d$. Then it follows from Lemma 3 and induction that

$$(2.32) \quad \begin{pmatrix} P_{jkm-1} & a_{km}P_{jkm-2} \\ Q_{jkm-1} & a_{km}Q_{jkm-2} \end{pmatrix} = \begin{pmatrix} P_{km-1} & a_{km}P_{km-2} \\ Q_{km-1} & a_{km}Q_{km-2} \end{pmatrix}^j \\ = \frac{(a+d)^{j-1}}{2^{j+1}c} \begin{pmatrix} (j(a-d) + a+d)2c & -j(a-d)^2 \\ 4jc^2 & (j(d-a) + a+d)2c \end{pmatrix}.$$

From this and (2.21) it follows that, for $0 \leq r \leq km-1$,

$$\begin{aligned} \frac{P_{jkm+r}}{Q_{jkm+r}} &= \frac{(j(a-d) + a+d)2cP_r - j(a-d)^2Q_r}{4jc^2P_r + (j(d-a) + a+d)2cQ_r} \\ &= \frac{((a-d) + (a+d)/j)2cP_r - (a-d)^2Q_r}{4c^2P_r + ((d-a) + (a+d)/j)2cQ_r}, \end{aligned}$$

and that

$$(2.33) \quad G(q) = \lim_{j \rightarrow \infty} \frac{P_{jkm+r}}{Q_{jkm+r}} = \frac{a-d}{2c} = \frac{P_{km-1} - a_{km}Q_{km-2}}{2Q_{km-1}}.$$

This holds whether or not $2cP_r - (a-d)Q_r = 0$, for any $r \in \{0, 1, \dots, km-1\}$.

Note that (2.10) implies that $|G(q)|$ depends only on s .

In the following lemmas the cases of equal and unequal eigenvalues are considered separately, with Lemmas 5 to 7 dealing with the case of equal eigenvalues.

Define $G_n(q) := P_n(q)/Q_n(q)$ and $H_n(q) := x^n/G_n(q)$, where η is as defined at (2.9).

For the case of equal eigenvalues, it follows from (2.25) that $T = P_{km-1} + a_{km}Q_{km-2} \neq 0$. Note also that the conditions at (2.10) imply that $P_{km-1} - a_{km}Q_{km-2} = K_1 - K_0 K_2$, a fixed integer depending only on s .

In the following lemmas a sequence of positive integers N_1, \dots, N_{10} and a sequence of constants is defined. These integers and constants will depend only on the constants K_0, K_1, K_2 and K_3 described at (2.10). As such, they will depend only on s . To avoid repetition throughout the lemmas, we state here that these integers are chosen to satisfy $N_1 < N_3 < N_4 < N_7 < N_8$ and $N_2 < N_5 < N_6 < N_9 < N_{10}$. For Lemmas 5 to 7, we assume that $\lambda_1 = \lambda_2$.

Lemma 5. *There exist positive constants D_1, D_2, D_3, D_4, D_5 and D_6 , each depending only on s , such that if $j \geq 1$, then*

$$(2.34) \quad |Q_{jkm-1}| = j D_2 D_1^j.$$

There exists a positive integer N_1 , depending only on s , such that if $j \geq N_1$, then

$$(2.35) \quad D_3 j D_1^j \leq |Q_{jkm-2}| \leq D_4 j D_1^j$$

and

$$(2.36) \quad D_5 \leq \left| \frac{Q_{jkm-1}}{Q_{jkm-2}} \right| \leq D_6.$$

Proof. To prove (2.34), we first equate entries at (2.32), using the fact that $\lambda_1 = \lambda_2 = (P_{km-1} + a_{km}Q_{km-2})/2$.

$$Q_{jkm-1} = j Q_{km-1} \lambda_1^{j-1},$$

(2.34) follows upon setting $D_1 = |\lambda_1|$ and $D_2 = |Q_{km-1}/\lambda_1|$, recalling the conditions at (2.10) and the facts noted at the end of (2.27).

Note for later use that, since M has determinant equal to a non-zero integer and has two equal eigenvalues, then $D_1 \geq 1$ so that $\lim_{j \rightarrow \infty} |Q_{jkm-1}| = \infty$. Inequality (2.36) then implies $\lim_{j \rightarrow \infty} |Q_{jkm-1}| = \infty$ also.

Statement (2.35) follows similarly from comparing corresponding matrix elements at (2.32), namely,

$$Q_{jkm-2} = \left(a_{km}Q_{km-2} - P_{km-1} + \frac{2\lambda_1}{j} \right) \frac{j\lambda_1^{j-1}}{2a_{km}}.$$

Set

$$D_4 = \frac{2|\lambda_1| + |a_{km}Q_{km-2} - P_{km-1}|}{2|a_{km}\lambda_1|}.$$

Take N_1 large enough so that $2|\lambda_1|/N_1 < |a_{km}Q_{km-2} - P_{km-1}|$ and then set

$$D_3 = \frac{|a_{km}Q_{km-2} - P_{km-1}| - 2|\lambda_1|/N_1}{2|a_{km}\lambda_1|}.$$

Statement (2.36) follows from (2.34) and (2.35), by taking $D_5 = D_2/D_4$ and $D_6 = D_2/D_3$. \square

Lemma 6. *There exists positive constants D_7, D_8, D_9, D_{10} and D_{11} and positive integers N_3 and N_4 , each depending only on s , such that, if $j \geq 1$, then*

$$(2.37) \quad |G(q) - G_{jkm-1}(q)| = \frac{D_7}{j},$$

if $j \geq N_3$, then

$$(2.38) \quad \frac{D_8}{j} \leq |G(q) - G_{jkm-2}(q)| \leq \frac{D_9}{j},$$

and if $j \geq N_4$ and $n = jkm - 1$ or $jkm - 2$, then

$$(2.39) \quad \frac{D_{10}}{j} \leq |H(q) - H_n(q)| \leq \frac{D_{11}}{j}.$$

Proof. Equations (2.32) and (2.33) give that

$$\begin{aligned} & |G(q) - G_{jkm-1}(q)| \\ &= \left| \frac{P_{km-1} - a_{km}Q_{km-2}}{2Q_{km-1}} - \frac{(j+1)P_{km-1} - (j-1)a_{km}Q_{km-2}}{2jQ_{km-1}} \right| \\ &= \left| \frac{P_{km-1} + a_{km}Q_{km-2}}{2jQ_{km-1}} \right|. \end{aligned}$$

Set

$$D_7 = \left| \frac{P_{km-1} + a_{km}Q_{km-2}}{2Q_{km-1}} \right|.$$

Note that $D_7 \neq 0$, since $D_7 = |\lambda_1/Q_{km-1}| \neq 0$. Next, from (2.32) and (2.33) we find that

$$\begin{aligned} & |G(q) - G_{jkm-2}(q)| \\ &= \left| \frac{P_{km-1} - a_{km}Q_{km-2}}{2Q_{km-1}} + \frac{j(P_{km-1} - a_{km}Q_{km-2})^2}{2Q_{km-1}((j+1)a_{km}Q_{km-2} - (j-1)P_{km-1})} \right| \\ &= \left| \frac{-P_{km-1}^2 + a_{km}^2Q_{km-2}^2}{2Q_{km-1}(-(P_{km-1} + a_{km}Q_{km-2})/j + (P_{km-1} - a_{km}Q_{km-2}))} \right| \frac{1}{j}. \end{aligned}$$

Choose N_3 such that $|(P_{km-1} + a_{km}Q_{km-2})/N_3| < |(P_{km-1} - a_{km}Q_{km-2})|$ and set

$$D_8 = \frac{|-P_{km-1}^2 + a_{km}^2Q_{km-2}^2|}{2|Q_{km-1}||(|P_{km-1} + a_{km}Q_{km-2}| + |P_{km-1} - a_{km}Q_{km-2}|)}$$

and

$$D_9 = \frac{|-P_{km-1}^2 + a_{km}^2Q_{km-2}^2|}{2|Q_{km-1}||(|P_{km-1} - a_{km}Q_{km-2}| - |P_{km-1} + a_{km}Q_{km-2}|/N_3)}.$$

Note that neither D_8 or D_9 is zero, since $\lambda_1 = (P_{km-1} + a_{km}Q_{km-2})/2 \neq 0$ and $P_{km-1} - a_{km}Q_{km-2} \neq 0$ from the remark following (2.31).

Let $n = jkm - 1$ or $jkm - 2$ and set $M' = \max\{D_7, D_9\}$ and $m' = \min\{D_7, D_8\}$. Choose N_4 such that $|G(q)| > M'/N_4$ (Recall that $|G(q)| \neq 0$ and is constant for fixed s). Let $j \geq N_4$. Then

$$|H(q) - H_n(q)| = \left| \frac{q^n}{G(q)} - \frac{q^n}{G_n(q)} \right| = \left| \frac{G(q) - G_n(q)}{G(q)G_n(q)} \right|.$$

Then

$$\frac{m'}{j |G(q)G_n(q)|} \leq |H(q) - H_n(q)| \leq \frac{M'}{j |G(q)G_n(q)|}.$$

By the definition of M' and the choice of N_4 , it follows that

$$\frac{m'}{j |G(q)| (|G(q)| + M')} \leq |H(q) - H_n(q)| \leq \frac{M'}{j |G(q)| (|G(q)| - M'/N_4)}.$$

Set

$$D_{10} = \frac{m'}{|G(q)| (|G(q)| + M')}, \quad D_{11} = \frac{M'}{|G(q)| (|G(q)| - M'/N_4)}.$$

The constants D_{10} and D_{11} depend only on s , since $|G(q)|$, m' , M' and N_4 depend only on s . \square

Lemma 7. *Let y be another point on the unit circle. There exist positive constants D_{13} , D_{14} and D_{15} and positive integers N_7 and N_8 , each depending only on s , such that if $j \geq N_7$ and $n = jkm - 1$ or $jkm - 2$, and*

$$P_n(y) = P_n(q) + \epsilon_1, \quad Q_n(y) = Q_n(q) + \epsilon_2, \quad \epsilon = \max\{|\epsilon_1|, |\epsilon_2|\} < 1/2,$$

then

$$(2.40) \quad |G_n(y) - G_n(q)| \leq \frac{D_{13}\epsilon}{j};$$

if $j \geq N_8$ and the angle between q and y (measured from the origin) is less than $\pi/(2|j|)$, then

$$(2.41) \quad |H_n(y) - H_n(q)| < D_{14}|q - y| + D_{15}\frac{\epsilon}{j}$$

and

$$(2.42) \quad |H_n(y) - H(q)| \leq D_{14}|q - y| + D_{15}\frac{\epsilon}{j} + \frac{D_{11}}{j}.$$

Proof. Let

$$D_{12} = \max\{D_7, D_9\}, \quad D'_2 = \min\{D_2, D_3\}, \quad N_7 \geq \left\lceil \frac{1}{D'_2} \right\rceil.$$

Set $D'_{13} = 3 + |G(q)| + D_{12}/N_7$ and set $D_{13} = 2D'_{13}/D'_2$. Choose N_8 such that

$$\min \left\{ |G(q)| - \frac{D_{12}}{N_8}, |G(q)| - \frac{D_{12}}{N_8} - \frac{D_{13}}{2N_8} \right\} \geq \frac{|G(q)|}{2}.$$

From the fact that $D_1 \geq 1$ together with (2.34) and (2.35), it follows that

$$|Q_n| \geq D'_2 j D_1^j \geq D'_2 j.$$

Let $j \geq N_7$. Then $|Q_n| - 1/2 > D'_2 j - 1/2 \geq D'_2 j/2$ and

$$\begin{aligned} |G_n(y) - G_n(q)| &= \left| \frac{P_n(y)}{Q_n(y)} - \frac{P_n(q)}{Q_n(q)} \right| = \left| \frac{\epsilon_1 Q_n(q) - \epsilon_2 P_n(q)}{Q_n(q)(Q_n(q) + \epsilon_2)} \right| \\ &\leq \frac{|\epsilon_1 - \epsilon_2|}{|Q_n(q) + \epsilon_2|} + \frac{|\epsilon_2| |P_n(q) - Q_n(q)|}{|Q_n(q)| |Q_n(q) + \epsilon_2|} \\ &= \frac{|\epsilon_1 - \epsilon_2|}{|Q_n(q) + \epsilon_2|} + \frac{|\epsilon_2| |G_n(q) - 1|}{|Q_n(q) + \epsilon_2|} \\ &\leq \frac{2\epsilon}{||Q_n(q)| - \epsilon|} + \frac{\epsilon (|G(q)| + D_{12}/j + 1)}{||Q_n(q)| - \epsilon|} \\ &= \frac{\epsilon (|G(q)| + D_{12}/j + 3)}{||Q_n(q)| - \epsilon|} \\ &\leq \frac{D'_{13} \epsilon}{||Q_n(q)| - 1/2|} \leq \frac{2D'_{13} \epsilon}{D'_2 j} = \frac{D_{13} \epsilon}{j}. \end{aligned}$$

Here we have used (2.37), (2.38), the bounds on ϵ_1 and ϵ_2 and the inequality relating $|Q_n|$ and j above.

Similarly, if $j \geq N_8$, then

$$\begin{aligned} |H_n(y) - H_n(q)| &= \left| \frac{y^n}{G_n(y)} - \frac{q^n}{G_n(q)} \right| \\ &= \left| \frac{G_n(q)(y^n - q^n) + q^n(G_n(q) - G_n(y))}{G_n(q)G_n(y)} \right| \\ &\leq \frac{2|\eta||q - y|}{|G_n(y)|} + \frac{|G_n(q) - G_n(y)|}{|G_n(q)||G_n(y)|} \\ &\leq \frac{2|\eta||q - y|}{||G_n(q)| - D_{13} \epsilon/j|} + \frac{D_{13} \epsilon/j}{|G_n(q)||G_n(q)| - D_{13} \epsilon/j}. \end{aligned}$$

Here we have used (2.40) and the fact that the angle between q and y (measured from the origin) is less than $\pi/(2|\eta|)$ implies that $|y^n - q^n| \leq 2|\eta||q - y|$ (This last inequality follows since the stated bound implies $(q/y)^n$ lies in the first or fourth quadrant and the fact that in these quadrants,

chordal distance from 1 is less than arc distance, which in turn is less than twice the chordal distance). From (2.37) and (2.38), it follows that

$$\begin{aligned}
(2.43) \quad |H_n(y) - H_n(q)| &\leq \frac{2|\eta||q - y|}{\|G(q)\| - (D_{12} + D_{13}\epsilon)/j} \\
&\quad + \frac{D_{13}\epsilon/j}{\|G(q)\| - D_{12}/j} \frac{\|G(q)\| - (D_{12} + D_{13}\epsilon)/j}{\|G(q)\| - (D_{12} + D_{13}\epsilon)/j} \\
&\leq \frac{2|\eta||q - y|}{\|G(q)\| - (D_{12} + D_{13}/2)/N_8} \\
&\quad + \frac{D_{13}\epsilon/j}{\|G(q)\| - D_{12}/N_8} \frac{\|G(q)\| - (D_{12} + D_{13}/2)/N_8}{\|G(q)\| - (D_{12} + D_{13}/2)/N_8} \\
(2.44) \quad &\leq \frac{4|\eta|}{|G(q)|} |q - y| + \frac{4D_{13}\epsilon}{|G(q)|^2 j}.
\end{aligned}$$

Set $D_{14} = \max\{4/|G(q)|, 4|\eta|/|G(q)|\}$ and $D_{15} = 4D_{13}/|G(q)|^2$.

Statement (2.42) follows from (2.41) and (2.39). \square

In the following three lemmas we assume $|\lambda_1| > |\lambda_2|$.

Lemma 8. *There exist positive constants $C_1, C_2, C_3, C_4, C_5, C_6$ and C_7 , and a positive integer N_2 , each depending only on s , such that, if $j \geq 1$, then*

$$(2.45) \quad C_2 C_1^j < |Q_{jkm-1}| < C_3 C_1^j;$$

and if $j \geq N_2$, then

$$(2.46) \quad C_4 C_1^j \leq |Q_{jkm-2}| \leq C_5 C_1^j$$

and

$$(2.47) \quad C_6 \leq \frac{|Q_{jkm-1}|}{|Q_{jkm-2}|} \leq C_7.$$

Proof. Let $\lambda_1, \lambda_2, x_1$ and y_1 be as defined at (2.25), (2.26) and (2.27). Then

$$\begin{pmatrix} P_{km-1} & a_{km}P_{km-2} \\ Q_{km-1} & a_{km}Q_{km-2} \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ 1 & 1 \end{pmatrix}^{-1}.$$

From Lemma 3 it follows that

$$\begin{aligned}
(2.48) \quad &\begin{pmatrix} P_{jkm-1} & a_{km}P_{jkm-2} \\ Q_{jkm-1} & a_{km}Q_{jkm-2} \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^j & 0 \\ 0 & \lambda_2^j \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ 1 & 1 \end{pmatrix}^{-1} \\
&= \frac{1}{x_1 - y_1} \begin{pmatrix} x_1\lambda_1^j - y_1\lambda_2^j & -x_1y_1(\lambda_1^j - \lambda_2^j) \\ \lambda_1^j - \lambda_2^j & x_1\lambda_2^j - y_1\lambda_1^j \end{pmatrix}.
\end{aligned}$$

Thus

$$Q_{jkm-1} = \frac{1 - (\lambda_2/\lambda_1)^j}{x_1 - y_1} \lambda_1^j.$$

Statement (2.45) follows with $C_1 = |\lambda_1|$ and

$$C_2 = \frac{1 - |\lambda_2/\lambda_1|}{|x_1 - y_1|} \quad \text{and} \quad C_3 = \frac{1 + |\lambda_2/\lambda_1|}{|x_1 - y_1|}.$$

Note that, since M has a non-zero integral determinant and $|\lambda_1| > |\lambda_2|$, $C_1 = |\lambda_1| > 1$.

Similarly,

$$Q_{jkm-2} = \frac{-y_1 \lambda_1^j}{a_{km}(x_1 - x_2)} \left(1 - \frac{x_1}{y_1} \left(\frac{\lambda_2}{\lambda_1} \right)^j \right).$$

Choose N_2 large enough so that

$$\left| \frac{x_1}{y_1} \right| \left| \frac{\lambda_2}{\lambda_1} \right|^{N_2} < 1$$

and then take

$$C_4 = \left| \frac{y_1}{a_{km}(x_1 - y_1)} \right| \left(1 - \left| \frac{x_1}{y_1} \right| \left| \frac{\lambda_2}{\lambda_1} \right|^{N_2} \right)$$

and

$$C_5 = \left| \frac{y_1}{a_{km}(x_1 - y_1)} \right| \left(1 + \left| \frac{x_1}{y_1} \right| \left| \frac{\lambda_2}{\lambda_1} \right| \right).$$

Note that equation (2.27) and the fact that none of a_{km} , Q_{km-1} and P_{km-2} is zero ensure that $x_1, y_1 \neq 0$, and hence that $C_4, C_5 \neq 0$. Clearly, for $j \geq N_2$,

$$C_6 := \frac{C_2}{C_5} \leq \left| \frac{Q_{jkm-1}}{Q_{jkm-2}} \right| \leq \frac{C_3}{C_4} =: C_7.$$

Note that, by the remarks following (2.27), all of these constants depend only on s . Note also that the condition $|\lambda_1| > 1$ implies $\lim_{j \rightarrow \infty} |Q_{jkm-1}| = \lim_{j \rightarrow \infty} |Q_{jkm-2}| = \infty$. □

Lemma 9. *There exist positive constants $C_8 < 1$, $C_9, C_{10}, C_{11}, C_{12}, C_{13}$ and C_{14} and positive integers N_5 and N_6 , each depending only on s , such that, if $j \geq 1$, then*

$$(2.49) \quad C_9 C_8^j \leq |G(q) - G_{jkm-1}(q)| \leq C_{10} C_8^j;$$

if $j \geq N_5$, then

$$(2.50) \quad C_{11} C_8^j \leq |G(q) - G_{jkm-2}(q)| \leq C_{12} C_8^j;$$

and if $j \geq N_6$ and $n = jkm - 1$ or $jkm - 2$, then

$$(2.51) \quad C_{13}C_8^j \leq |H(q) - H_n(q)| \leq C_{14}C_8^j.$$

Proof. From (2.48) it can be seen that $G(q)$ converges to x_1 (and thus, from (2.10) and (2.26), that $|G(q)|$ depends only on s) so that

$$|G(q) - G_{jkm-1}(q)| = \left| x_1 - \frac{x_1\lambda_1^j - y_1\lambda_2^j}{\lambda_1^j - \lambda_2^j} \right| = \frac{|x_1 - y_1|}{|1 - (\lambda_2/\lambda_1)^j|} \left| \frac{\lambda_2}{\lambda_1} \right|^j.$$

Set $C_8 = |\lambda_2/\lambda_1| < 1$, $C_9 = |x_1 - y_1|/(1 + C_8)$ and $C_{10} = |x_1 - y_1|/(1 - C_8)$, and (2.49) follows. Note that $x_1 \neq y_1$ (else the eigenvalues would be equal), so that C_9 and C_{10} are non-zero.

Next, choose N_5 large enough so that $C_8^{N_5} < |y_1/x_1|$, and consider $j \geq N_5$. Thus,

$$|G(q) - G_{jkm-2}(q)| = \left| x_1 - \frac{x_1y_1(\lambda_1^j - \lambda_2^j)}{-x_1\lambda_2^j + y_1\lambda_1^j} \right| = \frac{|x_1 - y_1|}{|y_1/x_1 - (\lambda_2/\lambda_1)^j|} \left| \frac{\lambda_2}{\lambda_1} \right|^j.$$

Set

$$C_{11} = \frac{|x_1 - y_1|}{|y_1/x_1| + C_8}, \quad C_{12} = \frac{|x_1 - y_1|}{|y_1/x_1| - C_8^{N_5}},$$

and (2.50) follows.

Finally, let $n = jkm - 1$ or $jkm - 2$, set $m' = \min\{C_9, C_{11}\}$ and $M' = \max\{C_{10}, C_{12}\}$. Choose N_6 such that $|G(q)| > M' C_8^{N_6}$. Let $j \geq N_6$. Then

$$|H(q) - H_n(q)| = \left| \frac{q^n}{G(q)} - \frac{q^n}{G_n(q)} \right| = \left| \frac{G(q) - G_n(q)}{G(q)G_n(q)} \right|.$$

From the definitions of m' and M' , and the choice of N_6 , it follows that

$$\frac{m' C_8^j}{|G(q)| (|G(q)| + M')} \leq |H(q) - H_n(q)| \leq \frac{M' C_8^j}{|G(q)| (|G(q)| - M' C_8^{N_6})}$$

Set

$$C_{13} = \frac{m'}{|G(q)| (|G(q)| + M')}, \quad C_{14} = \frac{M'}{|G(q)| (|G(q)| - M' C_8^{N_6})},$$

and (2.51) follows. \square

Lemma 10. *Let y be a another point on the unit circle. There exist positive constants C_{15} , C_{16} and C_{17} and positive integers N_9 and N_{10} , each depending only on s , such that if $j \geq N_9$, $n = jkm - 1$ or $jkm - 2$. and,*

$$P_n(y) = P_n(q) + \epsilon_1, \quad Q_n(y) = Q_n(q) + \epsilon_2, \quad \text{with } \epsilon = \max\{|\epsilon_1|, |\epsilon_2|\} < 1/2.$$

then

$$(2.52) \quad |G_n(y) - G_n(q)| \leq \frac{C_{15} \epsilon}{C_1^j};$$

If $j \geq N_{10}$, $n = jkm - 1$ or $jkm - 2$ and the angle between q and y (measured from the origin) is less than $\pi/(2|\eta|)$, then

$$(2.53) \quad |H_n(y) - H_n(q)| < C_{16}|q - y| + C_{17} \frac{\epsilon}{C_1^j}$$

and

$$(2.54) \quad |H_n(y) - H(q)| \leq C_{16}|q - y| + C_{17} \frac{\epsilon}{C_1^j} + C_{14} C_8^j.$$

Proof. Let

$$C'_{10} = \max\{C_{10}, C_{12}\}, \quad C'_2 = \min\{C_2, C_4\} \quad \text{and} \quad N_9 \geq \frac{-\log(C'_2)}{\log(C_1)}.$$

Set $C'_{15} = 3 + |G(q)| + C'_{10} C_8^{N_9}$ and $C_{15} = 2C'_{15}/C'_2$. Choose N_{10} such that

$$\min \left\{ |G(q)| - C'_{10} C_8^{N_{10}}, \quad |G(q)| - C'_{10} C_8^{N_{10}} - \frac{C_{15}}{2C_1^{N_{10}}} \right\} \geq \frac{|G(q)|}{2}.$$

Let $j \geq N_9$. The inequalities at (2.45) and (2.46) imply that $|Q_n| \geq C'_2 C_1^j$. The condition on N_9 implies that, if $j \geq N_9$, then $|Q_n| - 1/2 > C'_2 C_1^j/2$. By similar reasoning to that used in the proof of (2.40), we find that

$$\begin{aligned} |G_n(y) - G_n(q)| &\leq \frac{|\epsilon_1 - \epsilon_2|}{|Q_n(q) + \epsilon_2|} + \frac{|\epsilon_2||G_n(q) - 1|}{|Q_n(q) + \epsilon_2|} \\ &\leq \frac{2\epsilon}{||Q_n(q)| - \epsilon|} + \frac{\epsilon \left| |G(q)| + C'_{10} C_8^j + 1 \right|}{||Q_n(q)| - \epsilon|} \\ &\leq \frac{C'_{15} \epsilon}{||Q_n(q)| - 1/2|} \leq \frac{2C'_{15} \epsilon}{C'_2 C_1^j} = \frac{C_{15} \epsilon}{C_1^j}. \end{aligned}$$

Here we have used (2.49), (2.50), the bounds on ϵ_1 and ϵ_2 in the statement of the lemma and the inequality relating $|Q_n|$ and C_1^j above.

Let $j \geq N_{10}$. As in the case where $\lambda_1 = \lambda_2$,

$$\begin{aligned} |H_n(y) - H_n(q)| &\leq \frac{2|\eta||q - y|}{|G_n(y)|} + \frac{|G_n(q) - G_n(y)|}{|G_n(q)||G_n(y)|} \\ &\leq \frac{2|\eta||q - y|}{\left| |G_n(q)| - C_{15} \epsilon / C_1^j \right|} + \frac{C_{15} \epsilon / C_1^j}{|G_n(q)| \left| |G_n(q)| - C_{15} \epsilon / C_1^j \right|}. \end{aligned}$$

Here we have used (2.52) and once again the fact that the angle between q and y , measured from the origin, is less than $\pi/(2|\eta|)$ implies that $|y^n - q^n| \leq$

$2|\eta||q - y|$ (See the explanation before (2.43)). Using (2.49) and (2.50), it follows that,

$$\begin{aligned}
|H_n(y) - H_n(q)| &\leq \frac{2|\eta||q - y|}{\left| |G(q)| - C'_{10}C_8^j - C_{15}\epsilon/C_1^j \right|} \\
&\quad + \frac{C_{15}\epsilon/C_1^j}{\left| |G(q)| - C'_{10}C_8^j \right| \left| |G(q)| - C'_{10}C_8^j - C_{15}\epsilon/C_1^j \right|} \\
&\leq \frac{2|\eta||q - y|}{\left| |G(q)| - C'_{10}C_8^{N_{10}} - C_{15}\epsilon/C_1^{N_{10}} \right|} \\
&\quad + \frac{C_{15}\epsilon/C_1^j}{\left| |G(q)| - C'_{10}C_8^{N_{10}} \right| \left| |G(q)| - C'_{10}C_8^{N_{10}} - C_{15}/(2C_1^{N_{10}}) \right|} \\
&\leq \frac{4|\eta|}{|G(q)|} |q - y| + \frac{4C_{15}}{|G(q)|^2} \frac{\epsilon}{C_1^j}.
\end{aligned}$$

Set $C_{16} = \max\{4|\eta|/|G(q)|, 4/|G(q)|\}$ and $C_{17} = 4C_{15}/|G(q)|^2$. Statement (2.54) follows from (2.53) and (2.51). \square

Lemma 11. *There exists an uncountable set of points on the unit circle such that, if y is one of these points, then there exist two increasing sequences of integer, $\{n_i\}_{i=1}^\infty$ and $\{m_i\}_{i=1}^\infty$ say, such that*

$$\begin{aligned}
\lim_{i \rightarrow \infty} H_{n_i}(y) &= \lim_{i \rightarrow \infty} H_{n_i-1}(y) = H_a, \\
\lim_{i \rightarrow \infty} H_{m_i}(y) &= \lim_{i \rightarrow \infty} H_{m_i-1}(y) = H_b,
\end{aligned}$$

for some $a, b \in \{1, 2, \dots, N_G\}$, where $a \neq b$.

Proof. If $\lambda_1 = \lambda_2$, we set $N' = N_8$. If $|\lambda_1| > |\lambda_2|$, we set $N' = N_{10}$. With the notation of Theorem 3, let $t \in S^\circ$ and set $y = \exp(2\pi it)$. Let c_{f_n}/d_{f_n} be one of the infinitely many approximants in the continued fraction expansion of t satisfying (2.13) and (2.14), and set $x_n = \exp(2\pi i c_{f_n}/d_{f_n})$, so that x_n is a primitive d_{f_n} -th root of unity and $H(x_n) = H_a$. Let $\gamma(n)$ be as defined at (2.18). We use, in turn, the fact that chord length is shorter than arc length, a standard bound on the absolute value of the difference between a real number and an approximant in its continued fraction expansion, and (2.14), we find that

$$(2.55) \quad |x_n - y| < 2\pi \left| t - \frac{c_{f_n}}{d_{f_n}} \right| < \frac{2\pi}{d_{f_n}^2 e_{f_n+1}} < \frac{1}{d_{f_n}^2 \gamma(k N' d_{f_n}^2)}.$$

Let $n' = k N' d_{f_n}^2 - 1$ or $k N' d_{f_n}^2 - 2$. By (2.16), (2.17), (2.18), and (2.55) it follows that

$$(2.56) \quad |P_{n'}(x_n) - P_{n'}(y)| \leq \gamma(n') |x_n - y| < \frac{1}{d_{f_n}^2},$$

and similarly

$$(2.57) \quad |Q_{n'}(x_n) - Q_{n'}(y)| \leq \frac{1}{d_{f_n}^2}.$$

If $\lambda_1 = \lambda_2$, then by (2.42), with $\epsilon = 1/d_{f_n}^2$ and $j = N' d_{f_n}$ (so that $j \geq N_8$), we find that

$$(2.58) \quad |H_{n'}(y) - H_a| = |H_{n'}(y) - H(x_n)| \leq \frac{D_{14}}{d_{f_n}^2 \gamma(k N' d_{f_n}^2)} + \frac{D_{15}}{d_{f_n}^3 N'} + \frac{D_{11}}{N' d_{f_n}}.$$

If $|\lambda_1| \neq |\lambda_2|$ then (2.54) similarly implies that

$$(2.59) \quad |H_{n'}(y) - H_a| = |H_{n'}(y) - H(x_n)| \leq \frac{C_{16}}{d_{f_n}^2 \gamma(k N' d_{f_n}^2)} + \frac{C_{17}}{d_{f_n}^2 C_1^{N' d_{f_n}}} + C_{14} C_8^{N' d_{f_n}}.$$

Thus, in either case,

$$(2.60) \quad \lim_{n \rightarrow \infty} H_{k N' d_{f_n}^2 - 1}(y) = \lim_{n \rightarrow \infty} H_{k N' d_{f_n}^2 - 2}(y) = H_a.$$

Similarly,

$$(2.61) \quad \lim_{n \rightarrow \infty} H_{k N' d_{g_n}^2 - 1}(y) = \lim_{n \rightarrow \infty} H_{k N' d_{g_n}^2 - 2}(y) = H_b.$$

The set S^\diamond is uncountable because the conditions for membership require restrictions on only infinitely many of the partial quotients. One can easily construct a subset for which there is no restriction on a fixed infinite set of partial quotient. For each set of choices of positive integers for these partial quotients, one can choose other partial quotients so that the conditions for membership of S^\diamond are fulfilled. Since the collection of all such continued fractions is uncountable, S^\diamond is an uncountable set. Thus $Y_G = \{\exp(2\pi it) : t \in S^\diamond\}$ is an uncountable set. \square

Before proving Theorem 3, we show that Y_G has measure zero. We use the following lemma.

Lemma 12. ([10] pp. 140–141) *Let $f(m) > 1$ for $m = 1, 2, \dots$, and suppose that $\sum_{n=1}^{\infty} 1/f(m) < \infty$. Then the set $S^* = \{t \in (0, 1) : e_m(t) > f(m) \text{ infinitely often}\}$ has measure zero.*

Let $f(m) = 2\pi\gamma(k N' m^2)$, where $\gamma(n)$ is the function at (2.18). Since $\gamma(j) \geq j$ for $j \geq 1$, it follows that $f(n) \geq 2\pi n^2$ and thus that $\sum_{n=1}^{\infty} 1/f(n)$ converges. Since, for the regular continued fraction expansion of any real number, $d_i > i$ for $i \geq 4$, it follows that $d_i^2 \geq (i+1)^2$ for $i \geq 4$, and thus it is clear from (2.14) that the elements in S^\diamond satisfy $e_m(t) > f(m)$ infinitely often. Hence S^\diamond (and thus Y_G) is a set of measure zero.

Of course the actual set of points on the unit circle at which $G(q)$ does not converge generally might have measure larger than zero.

Proof of Theorem 3. Let y be any point in Y_G , where Y_G is defined in the proof of Lemma 11, and let t be the irrational in $(0, 1)$ for which $y = \exp(2\pi it)$. N' is defined in Lemma 11. If $\lambda_1 = \lambda_2$, we set $N_{1/2} = N_1$. If $|\lambda_1| > |\lambda_2|$, we set $N_{1/2} = N_2$.

Suppose $H(y)$ converges generally to $f \in \hat{\mathbb{C}}$ and that $\{v_n\}, \{w_n\}$ are two sequences such that

$$\lim_{n \rightarrow \infty} \frac{P_n + v_n P_{n-1}}{Q_n + v_n Q_{n-1}} = \lim_{n \rightarrow \infty} \frac{P_n + w_n P_{n-1}}{Q_n + w_n Q_{n-1}} = \frac{y^n}{f} := g.$$

Suppose first that $|g| < \infty$. By construction, there exist two infinite strictly increasing sequences of positive integers $\{n_i\}_{i=1}^{\infty}, \{m_i\}_{i=1}^{\infty} \subset \mathbb{N}$ such that

$$L_a := \frac{y^n}{H_a} = \lim_{i \rightarrow \infty} \frac{P_{n_i}(y)}{Q_{n_i}(y)} = \lim_{i \rightarrow \infty} \frac{P_{n_i-1}(y)}{Q_{n_i-1}(y)}$$

and

$$L_b := \frac{y^n}{H_b} = \lim_{i \rightarrow \infty} \frac{P_{m_i}(y)}{Q_{m_i}(y)} = \lim_{i \rightarrow \infty} \frac{P_{m_i-1}(y)}{Q_{m_i-1}(y)},$$

for some $a \neq b$, where $a, b \in \{1, 2, \dots, N_G\}$. Also by construction each n_i has the form $k N' d_{k_i}^2 - 1$, where d_{k_i} is some denominator convergent in the continued fraction expansion of t . A similar situation holds for each m_i . It can be further assumed that $L_a \neq g$, since $L_a \neq L_b$. For ease of notation write

$$\begin{aligned} P_{n_i}(y) &= P_{n_i}, & Q_{n_i}(y) &= Q_{n_i}, \\ P_{n_i-1}(y) &= P_{n_i-1}, & Q_{n_i-1}(y) &= Q_{n_i-1}. \end{aligned}$$

Write $P_{n_i} = Q_{n_i}(L_a + \epsilon_{n_i})$ and $P_{n_i-1} = Q_{n_i-1}(L_a + \delta_{n_i})$, where $\epsilon_{n_i} \rightarrow 0$ and $\delta_{n_i} \rightarrow 0$ as $i \rightarrow \infty$, so that

$$\frac{Q_{n_i}(L_a + \epsilon_{n_i}) + w_{n_i} Q_{n_i-1}(L_a + \delta_{n_i})}{Q_{n_i} + w_{n_i} Q_{n_i-1}} = g + \gamma_{n_i},$$

where $\gamma_{n_i} \rightarrow 0$ as $i \rightarrow \infty$. Thus

$$w_{n_i} + \frac{Q_{n_i}}{Q_{n_i-1}} = \frac{Q_{n_i}}{Q_{n_i-1}} \times \frac{\epsilon_{n_i} - \delta_{n_i}}{g - L_a + \gamma_{n_i} - \delta_{n_i}}.$$

Because of (2.36) or (2.47), the fact that each n_i has the form $k N' d_{k_i}^2 - 1$, where d_{k_i} is some denominator convergent in the continued fraction expansion of t and (2.57), it follows that Q_{n_i}/Q_{n_i-1} is absolutely bounded for $N' d_{k_i} > N_{1/2}$. Therefore the right hand side of the last equality tends to 0 as $i \rightarrow \infty$ and thus

$$(2.62) \quad w_{n_i} + Q_{n_i}/Q_{n_i-1} \rightarrow 0 \text{ as } n_i \rightarrow \infty.$$

Note that $|w_{n_i}| < \infty$ for all i sufficiently large, since $|Q_{n_i}/Q_{n_i-1}| < \infty$. Similarly,

$$(2.63) \quad v_{n_i} + Q_{n_i}/Q_{n_i-1} \rightarrow 0 \text{ as } n_i \rightarrow \infty.$$

By the (2.62), (2.63) and the triangle inequality,

$$\lim_{i \rightarrow \infty} |v_{n_i} - w_{n_i}| = 0.$$

Thus

$$\liminf_{n \rightarrow \infty} d(v_n, w_n) = 0.$$

Therefore $H(y)$ does not converge generally. The proof in the case where g is infinite is similar.

Since Y_G is uncountable, this proves the theorem. □

Remark: Clearly $H(y) = y^n/G(y)$ converges generally if and only if $G(y)$ converges generally.

We have the following corollary to Theorem 3.

Corollary 2. *For each of the continued fractions $K(q)$, $S_1(q)$, $S_2(q)$ and $S_3(q)$, there exists an uncountable set of points on the unit circle at which the continued fraction does not converge generally.*

$G(q)$	$K(q)$	$S_1(q)$	$S_2(q)$	$S_3(q)$
η	$1/5$	$1/8$	$1/2$	$1/3$
(s, d)	$(1, 5)$	$(1, 8)$	$(1, 8)$	$(1, 6)$
$H(q)$	$\frac{2 \exp(2\pi i r/5)}{1+\sqrt{5}}$	$\frac{1}{\sqrt{2} \exp(-\pi i r/4)}$	$\frac{1}{(1+\sqrt{2}) \exp(\pi i r)}$	$\frac{1}{2 \exp(4\pi i r/3)}$
k	1	2	2	1
f_1, \dots, f_k	x	$qx^2, qx + q^2x^2$	$qx^2 + q^2x^4, q^4x^4$	$x + x^2$
a_{km}	1	2	1	2
P_{km-1}	1	2	3	1
Q_{km-2}	0	1	1	0
Q_{km-1}	$q^{(m-1)/5}$	$\epsilon q^{(m^2-1)/8}$	$q^{(m-1)/2}$	$q^{(m-1)/3}$
P_{km-2}	$q^{(1-m)/5}$	$\epsilon q^{(m-1)^2/8}$	$q^{(m+1)/2}$	$q^{(2m+1)/3}$

TABLE 1

Proof. We use information contained in Table 1. In each case, $q = \exp(2\pi ir/m)$, a primitive m -th root of unity and $m \equiv s \pmod{d}$, where (s, d) is the pair of integers from (2.8). k is the integer and f_1, \dots, f_k are the polynomials from the definition of the continued fraction $G(q)$ at (2.2). $H(q) := q^\eta/G(q)$, where η is the rational in row one of the table.

Row three gives the value of $H(q)$, when $q = \exp(2\pi ir/m)$ as above. a_{km} is the km -th partial numerator in $G(q)$, as defined at (2.2).

The values in the first, third and last four rows come from the papers of Schur ([11]) and Zhang ([13]). The values of a_{km} can be determined from the continued fractions at (1.1) and (1.4) – (1.6). For the last two entries in the $S_1(q)$ column, $\epsilon = (-1)^{(m-1)/4}$, this notation being employed to make the table fit the width of the page.

We give the proof for $S_1(q)$ only, since the proof for each of the other continued fractions is almost identical. One can easily check that $S_1(q)$ has the form given at (2.2) and satisfies the condition at (2.3), with $k = 2$. From the table (or the paper of Zhang [13]), $S_1(q)$ satisfies (2.8) with $d = 8$ and $s = 1$. Likewise, (2.9) is satisfied with $\eta = 1/8$. Conditions (2.10) are satisfied with $K_0 = 2$, $K_1 = 2$, $K_2 = 1$ and $K_3 = K_4 = 1$ (when $m \equiv 1 \pmod{8}$). It is clear from row three of the table that (2.11) is satisfied, provided we choose $r \not\equiv u \pmod{8}$. The conditions required by the theorem are satisfied, and the result follows. \square

3. CONCLUDING REMARKS

In proving the existence of an uncountable set of points on the unit circle at which a q -continued fraction $G(q)$ does not converge in the general sense, our methods rely on knowing the behavior of the continued fraction at roots of unity and, if q is a primitive m -th root of unity, on the fact that the values of $a_{km}(q)$, $P_{km-1}(q)$, $Q_{km-2}(q)$ and $Q_{km-1}(q)P_{km-2}(q)$ are fixed for m belonging to certain arithmetic progressions (See (2.10)). Also important is the number η from (2.9) which leads to the continued fraction $H(q)$ taking only finitely many values at roots of unity. Such q -continued fractions appear to be quite special and it would be interesting to have a complete classification of them.

Our methods permit us to show the existence of a set of measure 0 at which each of the continued fractions diverges generally. We conjecture that each of these continued fractions diverges generally almost everywhere on the unit circle although at present we do not see how to prove this. It would be very interesting if a point on the unit circle which is not a root of unity could be exhibited at which any one of the continued fractions which are subject of Corollary 2 converged, in either the classical or general sense.

The most famous q -continued fraction after the Rogers-Ramanujan continued fraction is the Göllnitz-Gordon continued fraction, $GG(q)$ (see (2.4)). This continued fraction tends to the same limit as $S_2(q)$, for each q inside the unit circle, but the behaviour at roots of unity is slightly different. As

far as we are aware, its behaviour at roots of unity has not been studied. Based on computer investigations, it would seem that $GG(q)$ satisfies the conditions of Theorem 3 and thus that the Göllnitz-Gordon continued fraction diverges at uncountably many points on the unit circle. We hope to show this in a later paper.

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