

Some implications of Chu's $_{10}\psi_{10}$ extension of Bailey's ${}_6\psi_6$ summation formula

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Abstract

Lucy Slater used Bailey's ${}_6\psi_6$ summation formula to derive the Bailey pairs she used to construct her famous list of 130 identities of the Rogers-Ramanujan type.

In the present paper we apply the same techniques to Chu's $_{10}\psi_{10}$ generalization of Bailey's formula to produce quite general Bailey pairs. Slater's Bailey pairs are then recovered as special limiting cases of these more general pairs.

In re-examining Slater's work, we find that her Bailey pairs are, for the most part, special cases of more general Bailey pairs containing one or more free parameters. Further, we also find new general Bailey pairs (containing one or more free parameters) which are also implied by the ${}_6\psi_6$ summation formula.

Slater used the Jacobi triple product identity (sometimes coupled with the quintuple product identity) to derive her infinite products. Here we also use other summation formulae (including special cases of the ${}_6\psi_6$ summation formula and Jackson's ${}_6\phi_5$ summation formula) to derive some of our infinite products.

We use the new Bailey pairs, and/or the summation methods mentioned above, to give new proofs of some general series-product identities due to Ramanujan, Andrews and others. We also derive a new general series-product identity, one which may be regarded as a partner to one of the Ramanujan identities. We also find new transformation formulae between basic hypergeometric series, new identities of Rogers-Ramanujan type, and new false theta series identities. Some of these latter are a kind of "hybrid" in that one side of the identity consists a basic hypergeometric series, while the other side is formed from a theta product multiplied by a false theta series. This type of identity appears to be new.

1 Introduction

Bailey's ${}_6\psi_6$ identity [5],

$${}_6\psi_6 \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e \end{matrix}; q, \frac{qa^2}{bcde} \right] = \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde; q)_\infty}, \quad (1.1)$$

is probably the most important summation formula for bilateral basic hypergeometric series, and has many applications to number theory and partitions - see Andrews' paper [2] for some examples. This summation formula was also the main tool used by Slater [16, 17] to derive Bailey pairs and, from these, her list of 130 identities of the Rogers-Ramanujan type.

Bailey's formula was extended by Shukla [14], and Shukla's formula was later further extended by Chu [6]. Let

$$\begin{aligned} K := K(a, b, c, d, e, u, v, q) = & [uv(\sigma_3 - a\sigma_1)(q\sigma_3 - a\sigma_1)a^2 + (1-q)uv(a^2 - \sigma_4)(a^2 - \sigma_2a + \sigma_4)a \\ & + (u+v)(a+uv)(\sigma_3 - a\sigma_1)(a^2 - q\sigma_4)a + (u^2 + a)(v^2 + a)(a^2 - \sigma_4)(a^2 - q\sigma_4)] \\ & \times \{(a^2 - bcde)(a^2 - bcdeq)(1-u)(a-u)(1-v)(a-v)\}^{-1}, \quad (1.2) \end{aligned}$$

where, for $1 \leq k \leq 4$, σ_k denotes the k -th elementary symmetric function in $\{b, c, d, e\}$. Then

Proposition 1.1. (Chu [6]) For complex numbers a, b, c, d and e satisfying $|a^2/bcdeq| < 1$, there holds the identity

$$\begin{aligned} & {}_{10}\psi_{10} \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e, qu, qa/u, qv, qa/v \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, u, a/u, v, a/v; q, \frac{a^2}{qbcde} \end{matrix} \right] \\ &= \sum_{n=0}^{\infty} \frac{(1-aq^{2n})(b, c, d, e, uq, vq, aq/u, aq/v; q)_n}{(1-a)(aq/b, aq/c, aq/d, aq/e, u, v, a/u, a/v; q)_n} \left(\frac{a^2}{qbcde} \right)^n \\ &\quad + \sum_{n=1}^{\infty} \frac{(1-q^{2n}/a)(b/a, c/a, d/a, e/a, uq/a, vq/a, q/u, q/v; q)_n}{(1-1/a)(q/b, q/c, q/d, q/e, 1/u, 1/v, u/a, v/a; q)_n} \left(\frac{a^2}{qbcde} \right)^n \\ &= K \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a; q)_{\infty}}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde; q)_{\infty}}. \end{aligned} \quad (1.3)$$

Upon letting $u \rightarrow \infty, v \rightarrow \infty$, we obtain Bailey's formula (1.1), while letting $u \rightarrow \infty$ recovers Shukla's [14] ${}_8\psi_8$ generalization of Bailey's formula. For most values of the parameters, the expression for K is quite complicated, but we note in passing that the case $b = a/c$ results in considerable simplification.

Corollary 1.2. For complex numbers a, c, d and e satisfying $|a/deq| < 1$, there holds the identity

$$\begin{aligned} & {}_{10}\psi_{10} \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, a/c, c, d, e, qu, qa/u, qv, qa/v \\ \sqrt{a}, -\sqrt{a}, cq, qa/c, qa/d, qa/e, u, a/u, v, a/v; q, \frac{a}{qde} \end{matrix} \right] \\ &= \frac{(1-u/c)(1-cu/a)(1-v/c)(1-cv/a)}{(1-u/a)(1-u)(1-v/a)(1-v)} \frac{(aq, q, cq/d, cq/e, aq/cd, aq/ce, aq/de, q, q/a; q)_{\infty}}{(cq, aq/c, aq/d, aq/e, cq/a, q/c, q/d, q/e, qa/de; q)_{\infty}}. \end{aligned} \quad (1.4)$$

Since (1.1) was used by Slater to derive her Bailey pairs, it is natural to ask if Chu's generalization of Bailey's formula at (1.3) can be used similarly to produce any new interesting results. The present paper is in part an investigation of that question.

One observation we make is that most of Slater's Bailey pairs are special cases of more general Bailey pairs containing one or more free parameters - see Corollaries 2.8 and 2.13. Slater could have derived these more general pairs herself, but it would seem that she was primarily interested in the special cases which would lead to identities of the Rogers-Ramanujan type.

We also note that Slater used the Jacobi Triple Product identity to derive her infinite products. In the present paper we use other summation formulae, including special cases of the ${}_6\psi_6$ - and ${}_6\phi_5$ summation formulae, to derive some infinite products.

In the transformation at (2.5) below, Slater essentially employed three cases ($y, z \rightarrow \infty$ and $y = \pm\sqrt{aq}, z \rightarrow \infty$) to make the series containing the α_n sequences summable. In the present paper we also explore additional cases, in the process discovering some interesting new identities (see Section 3 below).

Our results include new proofs of some general series-product identities due to Ramanujan, Andrews and others, and also a new general series-product identity

$$\sum_{n=0}^{\infty} \frac{(q/z; q)_{n+1}(z; q)_n q^{n^2+n}}{(q; q)_{2n+1}} = \frac{(zq^2, q/z, q^3; q^3)_{\infty}}{(q; q)_{\infty}}, \quad (1.5)$$

which may be regarded as a partner to an identity equivalent to one recorded by Ramanujan in his lost notebook [4, p. 99, Entry 5.3.1 with $a = -z/q$ and $q \rightarrow \sqrt{q}$ throughout]:

$$\sum_{n=0}^{\infty} \frac{(q/z; q)_n (z; q)_n q^{n^2}}{(q; q)_{2n}} = \frac{(zq, q^2/z, q^3; q^3)_{\infty}}{(q; q)_{\infty}}. \quad (1.6)$$

We also find new transformation formulae between basic hypergeometric series, new identities of Rogers-Ramanujan type, and new false theta series identities. Some of these latter are a kind of "hybrid" in that one side of the identity consists a basic hypergeometric series, while the other side is formed from a theta product multiplied by a false theta series. For example,

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q^{2n+1})(q; q)_n^2} = \frac{\sum_{n=0}^{\infty} q^{6n^2+2r}(1-q^{8r+4})}{(q; q)_{\infty}} \quad (1.7)$$

This type of identity appears to be new.

We employ the usual notations:

$$\begin{aligned}(a; q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}), \\ (a_1, a_2, \dots, a_j; q)_n &:= (a_1; q)_n(a_2; q)_n\cdots(a_j; q)_n, \\ (a; q)_\infty &:= (1-a)(1-aq)(1-aq^2)\cdots, \text{ and} \\ (a_1, a_2, \dots, a_j; q)_\infty &:= (a_1; q)_\infty(a_2; q)_\infty\cdots(a_j; q)_\infty,\end{aligned}$$

For later use, we recall that a pair of sequences (α_n, β_n) that satisfy $\alpha_0 = 1$ and

$$\beta_n = \sum_{j=0}^n \frac{\alpha_j}{(q; q)_{n-j}(aq; q)_{n+j}}, \quad (1.8)$$

is termed a *Bailey pair relative to a*.

2 Bailey Pairs from Chu's Extension of the ${}_6\psi_6$ summation formula

2.1 General Bailey pairs

We first consider the case $e = a$ and $d = q^{-N}$ in Chu's formula. These same substitutions were made by Slater in (1.1), and gave rise to a quite general Bailey pair (see below), which in turn led to many new identities of Rogers-Ramanujan type. Let K be defined as at (1.2).

Theorem 2.1. *The pair of sequences (α_n, β_n) is a Bailey pair with respect to a, where*

$$\begin{aligned}\alpha_n &:= \frac{(q\sqrt{a}, -q\sqrt{a}, a, b, c, qu, qa/u, qv, qa/v; q)_n}{(\sqrt{a}, -\sqrt{a}, qa/b, qa/c, u, a/u, v, a/v, q; q)_n} \left(\frac{-a}{bc}\right)^n q^{n(n-3)/2}, \\ \beta_n &:= K_1(n) \frac{(aq/bc; q)_n}{(aq/b, aq/c, q; q)_n},\end{aligned} \quad (2.1)$$

where $K_1(n) := K(a, b, c, q^{-n}, a, u, v, q)$.

Proof. First set $e = a$, so that all the terms with negative index in the bilateral sum at (1.3) become zero. Next, for each non-negative integer n , set $d = q^{-n}$, so that the sum on the left side of (1.3) becomes a finite sum, with the summation index running from 0 to n . Simplify the resulting product on the right side of (1.3), and use the identity (see [7, (I.10), page 351])

$$(q^{-n}; q)_j = \frac{(q; q)_n q^{j(j-1)/2}}{(q; q)_{n-j} (-q^n)^j} \quad (2.2)$$

to modify the series side. The result then follows from (1.8), after some simple manipulations, noting also that $K_1(0) = 1$, so that the requirement $\alpha_0 = \beta_0 = 1$ is satisfied. \square

The expression for $K_1(n)$ above is generally quite complicated, and thus so also is the expression for the β_n . However, as was also the case for K above, if we set $b = a/c$ then $K_1(n)$ simplifies considerably. One can check, preferably using a computer algebra system, that

$$K(a, a/c, c, q^{-n}, a, u, v, q) = \begin{cases} 1, & n = 0, \\ \frac{(c-u)(c-v)(a-cu)(a-cv)}{c^2(a-u)(a-v)(1-u)(1-v)} \\ - \frac{(1-c)(c-a)(c-aq)(1-cq)uv}{c^2q(a-u)(1-u)(a-v)(1-v)}, & n = 1, \\ \frac{(c-u)(c-v)(a-cu)(a-cv)}{c^2(a-u)(a-v)(1-u)(1-v)}, & n > 1. \end{cases} \quad (2.3)$$

Upon letting $u \rightarrow \infty, v \rightarrow \infty$ in the Bailey pair in Theorem 2.1, we recover the following Bailey pair (with respect to a) due to Slater,

$$\alpha_n = \frac{(1-aq^{2n})(a, b, c; q)_n}{(1-a)(aq/b, aq/c, q; q)} \left(\frac{-a}{bc}\right)^n q^{(n^2+n)/2}, \quad (2.4)$$

$$\beta_n = \frac{(aq/bc; q)_n}{(aq/b, aq/c, q; q)_n},$$

which is implicitly contained in Equation (4.1) of [16].

It would appear that Slater's principal motivation was to prove identities of the Rogers-Ramanujan type, so this Bailey pair, and other general pairs mentioned below, were not stated explicitly by her in [16] and [17], where she instead listed many special cases of them. However, they could all have been easily derived by her, using the same methods she used to derive the special cases.

We remark in passing that all the Bailey pairs in Slater's **B**, **F** and **H** tables, as well as pairs **E(3)**, **E(6)** and **E(7)** (see [16, page 468]), are derived from the Bailey pair at (2.4).

Slater also showed [16, Equation (1.3) on page 462] that if (α_n, β_n) is a Bailey pair with respect to a , then, for non-zero complex numbers y and z ,

$$\sum_{n=0}^{\infty} (y, z; q)_n \left(\frac{aq}{yz}\right)^n \beta_n = \frac{(aq/y, aq/z; q)_{\infty}}{(aq, aq/yz; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(y, z; q)_n}{(aq/y, aq/z; q)_n} \left(\frac{aq}{yz}\right)^n \alpha_n. \quad (2.5)$$

A finite generalization of this identity is of course implied by Bailey's Lemma (see, for example, Theorem 12.2.3 in [3]), namely, if (α_n, β_n) is a Bailey pair with respect to a , and N is a non-negative integer, then

$$\sum_{n=0}^N \frac{(y, z, q^{-N}; q)_n}{(yzq^{-N}/a; q)_n} q^n \beta_n = \frac{(aq/y, aq/z; q)_N}{(aq, aq/yz; q)_N} \sum_{n=0}^N \frac{(y, z, q^{-N}; q)_n}{(aq/y, aq/z, aq^{1+N}; q)_n} \left(\frac{-aq^N}{yz}\right)^n q^{-n(n-3)/2} \alpha_n. \quad (2.6)$$

The identity at (2.5) is a particular case of the *Bailey Transform*: if $\beta_n = \sum_{r=0}^n \alpha_r u_{n-r} v_{n+r}$ and $\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}$, then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n.$$

In the present paper, for ease of notation we will refer to (2.5) as the *Bailey Transform*.

If we set $b = a/c$ in the Bailey pair from Theorem 2.1, and then substitute this pair into (2.5) and (2.6), we get the following unusual basic hypergeometric identities.

Corollary 2.2. *Let $N \geq 1$ be an integer. Then*

$$1 + \frac{(c-u)(c-v)(a-cu)(a-cv)}{c^2(a-u)(a-v)(1-u)(1-v)} \sum_{n=1}^N \frac{(y, z, q^{-N}; q)_n}{\left(cq, \frac{aq}{c}, \frac{yzq^{-N}}{a}; q\right)_n} q^n = \frac{a(c-a)(1-c)(1-y)(1-z)(1-q^N)uv}{c(a-u)(a-v)(1-u)(1-v)(yz-aq^N)} \\ + \frac{\left(\frac{aq}{y}, \frac{aq}{z}; q\right)_N}{\left(aq, \frac{aq}{yz}; q\right)_N} \sum_{n=0}^N \frac{\left(q\sqrt{a}, -q\sqrt{a}, a, \frac{a}{c}, c, qu, \frac{qa}{u}, qv, \frac{qa}{v}, y, z, q^{-N}; q\right)_n}{\left(\sqrt{a}, -\sqrt{a}, cq, \frac{qa}{c}, u, \frac{a}{u}, v, \frac{a}{v}, \frac{aq}{y}, \frac{aq}{z}, aq^{1+N}, q; q\right)_n} \left(\frac{aq^N}{yz}\right)^n; \quad (2.7)$$

$$1 + \frac{(c-u)(c-v)(a-cu)(a-cv)}{c^2(a-u)(a-v)(1-u)(1-v)} \sum_{n=1}^{\infty} \frac{(y, z; q)_n}{\left(cq, \frac{aq}{c}; q\right)_n} \left(\frac{aq}{yz}\right)^n = \frac{a(c-a)(1-c)(1-y)(1-z)uv}{c(a-u)(a-v)(1-u)(1-v)yz} \\ + \frac{\left(\frac{aq}{y}, \frac{aq}{z}; q\right)_{\infty}}{\left(aq, \frac{aq}{yz}; q\right)_{\infty}} \sum_{n=0}^{\infty} \frac{\left(q\sqrt{a}, -q\sqrt{a}, a, \frac{a}{c}, c, qu, \frac{qa}{u}, qv, \frac{qa}{v}, y, z; q\right)_n}{\left(\sqrt{a}, -\sqrt{a}, cq, \frac{qa}{c}, u, \frac{a}{u}, v, \frac{a}{v}, \frac{aq}{y}, \frac{aq}{z}, q; q\right)_n} \left(\frac{-a}{yz}\right)^n q^{n(n-1)/2}. \quad (2.8)$$

Note that setting $c = a$ in (2.7) gives the q -Pfaff-Saalschütz sum (see [7, page 355, II.12]), while setting $c = u$ gives the identity (for $N \geq 0$)

$$\sum_{n=0}^N \frac{\left(q\sqrt{a}, -q\sqrt{a}, a, qv, \frac{qa}{v}, y, z, q^{-N}; q\right)_n}{\left(\sqrt{a}, -\sqrt{a}, v, \frac{a}{v}, \frac{aq}{y}, \frac{aq}{z}, aq^{1+N}, q; q\right)_n} \left(\frac{aq^N}{yz}\right)^n = \left[1 + \frac{a(1-y)(1-z)(1-q^N)v}{(a-v)(1-v)(yz-aq^N)}\right] \frac{\left(aq, \frac{aq}{yz}; q\right)_N}{\left(\frac{aq}{y}, \frac{aq}{z}; q\right)_N}. \quad (2.9)$$

2.2 Mod 3 Bailey pairs

We continue to follow in Slater's footsteps, this time making the same substitutions in Chu's identity (1.3) that she did in (1.1) to produce the Bailey pairs in her **A** table. Let K be as defined at (1.2).

Theorem 2.3. (i) Let $K_0(n) := K(a, q^{-n}, q^{1-n}, q^{2-n}, a, u, v, q^3)$. Then the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = a$, where $\alpha_0 = \beta_0 = 1$, $\alpha_{3r\pm 1} = 0$, and

$$\alpha_{3r} = \frac{1 - aq^{6r}}{1 - a} \frac{(a, q^3 u, q^3 v, aq^3/u, aq^3/v; q^3)_r}{(u, v, a/u, a/v, q^3; q^3)_r} q^{\frac{9r^2-15r}{2}} (-a)^r \quad (2.10)$$

$$\beta_n = K_0(n) \frac{(a; q^3)_n}{(a; q)_{2n}(q; q)_n}.$$

Let $K'_0(n) := K(aq, q^{-n}, q^{1-n}, q^{2-n}, aq, u, v, q^3)$. Then

(ii) the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = a$, where $\alpha_0 = \beta_0 = 1$, $\alpha_{3r-1} = 0$, and

$$\alpha_{3r} = \frac{(aq, q^3 u, q^3 v, aq^4/u, aq^4/v; q^3)_r}{(u, v, aq/u, aq/v, q^3; q^3)_r} q^{\frac{9r^2-13r}{2}} (-a)^r \quad (2.11)$$

$$\alpha_{3r+1} = \frac{(aq, q^3 u, q^3 v, aq^4/u, aq^4/v; q^3)_r}{(u, v, aq/u, aq/v, q^3; q^3)_r} q^{\frac{9r^2-r}{2}+1} (-a)^{r+1}$$

$$\beta_n = K'_0(n) \frac{(aq; q^3)_n}{(aq; q)_{2n}(q; q)_n};$$

(iii) the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = a$, where $\alpha_0 = \beta_0 = 1$, $\alpha_{3r-1} = 0$, and

$$\alpha_{3r} = \frac{(aq, q^3 u, q^3 v, aq^4/u, aq^4/v; q^3)_r}{(u, v, aq/u, aq/v, q^3; q^3)_r} q^{\frac{9r^2-7r}{2}} (-a)^r \quad (2.12)$$

$$\alpha_{3r+1} = -\frac{(aq, q^3 u, q^3 v, aq^4/u, aq^4/v; q^3)_r}{(u, v, aq/u, aq/v, q^3; q^3)_r} q^{\frac{9r^2-7r}{2}} (-a)^r$$

$$\beta_n = K'_0(n) \frac{(aq; q^3)_n q^n}{(aq; q)_{2n}(q; q)_n}.$$

Proof. Set $e = a$ in (1.3), so that all the terms of negative index vanish. Then replace q with q^3 , set $b = q^{-n}$, $c = q^{1-n}$ and $d = q^{2-n}$. Then after some simple manipulations (1.3) becomes

$$\sum_{r=0}^{n/3} \frac{1 - aq^{6r}}{1 - a} \frac{(q^{-n}; q)_{3r}}{(aq^{n+1}; q)_{3r}} \frac{(a, q^3 u, q^3 v, aq^3/u, aq^3/v; q^3)_r}{(u, v, a/u, a/v, q^3; q^3)_r} (aq^{3n-6})^r = K_0(n) \frac{(a; q^3)_n (aq; q)_n}{(a; q)_{2n}}.$$

Apply (2.2) to the $(q^{-n}; q)_{3r}$ factor, divide both sides by $(aq; q)_n (q; q)_n$ to get

$$\sum_{r=0}^{n/3} \frac{1 - aq^{6r}}{1 - a} \frac{(-1)^r q^{(9r^2-15r)/2}}{(aq; q)_{n+3r}(q; q)_{n-3r}} \frac{(a, q^3 u, q^3 v, aq^3/u, aq^3/v; q^3)_r}{(u, v, a/u, a/v, q^3; q^3)_r} a^r = K_0(n) \frac{(a; q^3)_n}{(a; q)_{2n}(q; q)_n} \quad (2.13)$$

and (2.10) follows.

For the other two pairs, replace a with aq in (2.13), and (2.11) follows from the identity

$$\frac{(1 - aq^{6r+1})q^{(9r^2-13r)/2}}{(aq; q)_{n+3r+1}(q; q)_{n-3r}} = \frac{q^{(9r^2-13r)/2}}{(aq; q)_{n+3r}(q; q)_{n-3r}} - \frac{aq^{(9r^2-r)/2+1}}{(aq; q)_{n+3r+1}(q; q)_{n-3r-1}},$$

while (2.12) follows from the identity

$$\frac{(1 - aq^{6r+1})q^{(9r^2-13r)/2}}{(aq; q)_{n+3r+1}(q; q)_{n-3r}} = \frac{q^{-n} q^{(9r^2-7r)/2}}{(aq; q)_{n+3r}(q; q)_{n-3r}} - \frac{q^{-n} q^{(9r^2-7r)/2}}{(aq; q)_{n+3r+1}(q; q)_{n-3r-1}}.$$

□

Remark 2.4. $K_0(0) = K_0(1) = K_0(2) = 1$.

Theorem 2.5. Let $K_2(n) := K(q, q^{-n}, q^{1-n}, q^{2-n}, e, u, v, q^3)$ and suppose $e \neq 1/q$. Then

(i) the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = 1$, where $\alpha_0 = \beta_0 = 1$ and

$$\begin{aligned}\alpha_{3r} &= (-1)^r \left(q^{\frac{9r^2-11r}{2}} \frac{(e, q^3u, q^3v, q^4/u, q^4/v; q^3)_r}{(q^4/e, u, v, q/u, q/v; q^3)_r} e^r + q^{\frac{9r^2-5r}{2}} \frac{(e/q, q^2u, q^2v, q^3/u, q^3/v; q^3)_r}{(q^3/e, u/q, v/q, 1/u, 1/v; q^3)_r} e^r \right), \\ \alpha_{3r+1} &= (-1)^{r+1} q^{\frac{9r^2+r}{2}+1} \frac{(e, q^3u, q^3v, q^4/u, q^4/v; q^3)_r}{(q^4/e, u, v, q/u, q/v; q^3)_r} e^r, \\ \alpha_{3r-1} &= (-1)^{r+1} q^{\frac{9r^2-17r}{2}+1} \frac{(e/q, q^2u, q^2v, q^3/u, q^3/v; q^3)_r}{(q^3/e, u/q, v/q, 1/u, 1/v; q^3)_r} e^r, \\ \beta_n &= K_2(n) \frac{(q^2/e; q^3)_n}{(q; q)_{2n} (q^2/e; q)_n};\end{aligned}\tag{2.14}$$

(ii) the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = 1$, where $\alpha_0 = \beta_0 = 1$ and

$$\begin{aligned}\alpha_{3r} &= (-1)^r \left(q^{\frac{9r^2-5r}{2}} \frac{(e, q^3u, q^3v, q^4/u, q^4/v; q^3)_r}{(q^4/e, u, v, q/u, q/v; q^3)_r} e^r + q^{\frac{9r^2-11r}{2}} \frac{(e/q, q^2u, q^2v, q^3/u, q^3/v; q^3)_r}{(q^3/e, u/q, v/q, 1/u, 1/v; q^3)_r} e^r \right), \\ \alpha_{3r+1} &= (-1)^{r+1} q^{\frac{9r^2-5r}{2}} \frac{(e, q^3u, q^3v, q^4/u, q^4/v; q^3)_r}{(q^4/e, u, v, q/u, q/v; q^3)_r} e^r, \\ \alpha_{3r-1} &= (-1)^{r+1} q^{\frac{9r^2-11r}{2}} \frac{(e/q, q^2u, q^2v, q^3/u, q^3/v; q^3)_r}{(q^3/e, u/q, v/q, 1/u, 1/v; q^3)_r} e^r, \\ \beta_n &= K_2(n) \frac{(q^2/e; q^3)_n q^n}{(q; q)_{2n} (q^2/e; q)_n},\end{aligned}\tag{2.15}$$

(iii) the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = q$, where $\alpha_0 = \beta_0 = 1$ and

$$\begin{aligned}\alpha_{3r} &= (-1)^r \frac{1 - q^{6r+1}}{1 - q} q^{\frac{9r^2-11r}{2}} \frac{(e, q^3u, q^3v, q^4/u, q^4/v; q^3)_r}{(q^4/e, u, v, q/u, q/v; q^3)_r} e^r \\ \alpha_{3r+1} &= 0, \\ \alpha_{3r-1} &= (-1)^{r+1} \frac{1 - q^{6r-1}}{1 - q} q^{\frac{9r^2-17r}{2}+1} \frac{(e/q, q^2u, q^2v, q^3/u, q^3/v; q^3)_r}{(q^3/e, u/q, v/q, 1/u, 1/v; q^3)_r} e^r, \\ \beta_n &= K_2(n) \frac{(q^2/e; q^3)_n}{(q; q)_{2n} (q^2/e; q)_n},\end{aligned}\tag{2.16}$$

Remark 2.6. The condition $e \neq 1/q$ is necessary to ensure that $\beta_0 = 1$.

Proof. Replace q with q^3 in (1.3), and then set $a = q$, $b = q^{-n}$, $c = q^{1-n}$ and $d = q^{2-n}$. One easily checks that the right side simplifies to give

$$K_2(n) \frac{(q, q^2; q)_n (q^2/e; q^3)_n}{(q; q)_{2n} (q^2/e; q)_n}.$$

On the series side, the choices for the parameters force the series to terminate above and below, and we get, after some elementary manipulations, that the series becomes

$$\sum_{r=-\frac{n+1}{3}}^{\frac{n}{3}} \frac{1 - q^{6r+1}}{1 - q} \frac{(q^{-n}; q)_{3r}}{(q^{n+2}; q)_{3r}} \frac{(e, q^3u, q^3v, q^4/u, q^4/v; q^3)_r}{(q^4/e, u, v, q/u, q/v; q^3)_r} \left(\frac{q^{3n-4}}{e} \right)^r.$$

Next, we apply (2.2) to the $(q^{-n}; q)_{3r}$ factor above and rearrange terms to get

$$\sum_{r=-\frac{n+1}{3}}^{\frac{n}{3}} \frac{(1 - q^{6r+1})(-1)^r q^{(9r^2-11r)/2}}{(q; q)_{n+3r+1} (q; q)_{n-3r} e^r} \frac{(e, q^3u, q^3v, q^4/u, q^4/v; q^3)_r}{(q^4/e, u, v, q/u, q/v; q^3)_r} = \frac{K_2(n) (q^2/e; q^3)_n}{(q; q)_{2n} (q^2/e; q)_n}.$$

Noting that, for arbitrary non-zero y and z ,

$$\frac{(y; q)_{-r}}{(z; q)_{-r}} = \frac{(q/z; q)_r z^r}{(q/y; q)_r y^r},\tag{2.17}$$

we get that

$$\begin{aligned}
K_2(n) & \frac{(q^2/e; q^3)_n}{(q; q)_{2n}(q^2/e; q)_n} \\
& = \sum_{r=0}^{n/3} \frac{(1-q^{6r+1})(-1)^r q^{(9r^2-11r)/2} (e, q^3 u, q^3 v, q^4/u, q^4/v; q^3)_r}{(q; q)_{n+3r+1}(q; q)_{n-3r} e^r (q^4/e, u, v, q/u, q/v; q^3)_r} \\
& \quad + \sum_{r=1}^{\frac{n+1}{3}} \frac{(1-q^{-6r+1})(-1)^r q^{(9r^2-5r)/2} (e/q, q^2 u, q^2 v, q^3/u, q^3/v; q^3)_r}{(q; q)_{n-3r+1}(q; q)_{n+3r} e^r (q^3/e, u/q, v/q, 1/u, 1/v; q^3)_r} \\
& = \sum_{r=0}^{n/3} \left(\frac{q^{(9r^2-11r)/2}}{(q; q)_{n+3r}(q; q)_{n-3r}} - \frac{q^{(9r^2+r)/2+1}}{(q; q)_{n+3r+1}(q; q)_{n-3r-1}} \right) (-1)^r \frac{(e, q^3 u, q^3 v, q^4/u, q^4/v; q^3)_r}{(q^4/e, u, v, q/u, q/v; q^3)_r} \\
& \quad + \sum_{r=1}^{\frac{n+1}{3}} \left(\frac{q^{(9r^2-5r)/2}}{(q; q)_{n-3r}(q; q)_{n+3r}} - \frac{q^{(9r^2-17r)/2+1}}{(q; q)_{n-3r+1}(q; q)_{n+3r-1}} \right) (-1)^r \frac{(e/q, q^2 u, q^2 v, q^3/u, q^3/v; q^3)_r}{(q^3/e, u/q, v/q, 1/u, 1/v; q^3)_r},
\end{aligned}$$

where the last equality follows from the identities

$$\begin{aligned}
(1-q^{6r+1})q^{\frac{9r^2-11r}{2}} & = q^{\frac{9r^2-11r}{2}}(1-q^{n+3r+1}) - q^{\frac{9r^2+r}{2}+1}(1-q^{n-3r}), \\
(1-q^{-6r+1})q^{\frac{9r^2-5r}{2}} & = q^{\frac{9r^2-5r}{2}}(1-q^{n-3r+1}) - q^{\frac{9r^2-17r}{2}+1}(1-q^{n+3r}).
\end{aligned} \tag{2.18}$$

That (2.14) gives a Bailey pair now follows from the definition of a Bailey pair at (1.8), also noting that $K_2(0) = K_2(1) = 1$.

If, instead of using the identities at (2.18), we use the identities

$$\begin{aligned}
(1-q^{6r+1})q^{\frac{9r^2-11r}{2}} & = q^{\frac{9r^2-5r}{2}-n}((1-q^{n+3r+1}) - (1-q^{n-3r})), \\
(1-q^{-6r+1})q^{\frac{9r^2-5r}{2}} & = q^{\frac{9r^2-11r}{2}-n}((1-q^{n-3r+1}) - (1-q^{n+3r})),
\end{aligned} \tag{2.19}$$

we get that the pair at (2.15) is a Bailey pair.

The proof of (2.16) follows from the right side of the first equality following (2.17), upon setting $(q; q)_{n+3r+1}(q; q)_{n-3r} = (1-q)(q^2; q)_{n+3r}(q; q)_{n-3r}$ in the first sum and $(q; q)_{n-3r+1}(q; q)_{n+3r} = (1-q)(q; q)_{n-3r+1}(q^2; q)_{n+3r-1}$ in the second sum. \square

If we begin by setting $a = q^2$ instead of $a = q$, but keep the same choices $b = q^{-n}$, $c = q^{1-n}$ and $d = q^{2-n}$ in (1.3), then we get Theorems 2.7 following.

Theorem 2.7. *Let $K_3(n) := K(q^2, q^{-n}, q^{1-n}, q^{2-n}, e, u, v, q^3)$. Then*

(i) *the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = q$, where $\alpha_0 = \beta_0 = 1$ and*

$$\begin{aligned}
\alpha_{3r} & = (-1)^r q^{\frac{9r^2-7r}{2}} \frac{(e, q^3 u, q^3 v, q^5/u, q^5/v; q^3)_r}{(q^5/e, u, v, q^2/u, q^2/v; q^3)_r} e^r, \\
\alpha_{3r+1} & = (-1)^{r+1} q^{\frac{9r^2+5r}{2}+2} \frac{(e, q^3 u, q^3 v, q^5/u, q^5/v; q^3)_r}{(q^5/e, u, v, q^2/u, q^2/v; q^3)_r} e^r + (-1)^r q^{\frac{9r^2-r}{2}-3} \frac{(e/q^2, qu, qv, q^3/u, q^3/v; q^3)_{r+1}}{(q^3/e, u/q^2, v/q^2, 1/u, 1/v; q^3)_{r+1}} e^{r+1}, \\
\alpha_{3r-1} & = (-1)^r q^{\frac{9r^2-7r}{2}} \frac{(e/q^2, qu, qv, q^3/u, q^3/v; q^3)_r}{(q^3/e, u/q^2, v/q^2, 1/u, 1/v; q^3)_r} e^r, \\
\beta_n & = K_3(n) \frac{(q^4/e; q^3)_n}{(q^2; q)_{2n}(q^3/e; q)_n};
\end{aligned} \tag{2.20}$$

(ii) *the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = q$, where $\alpha_0 = \beta_0 = 1$ and*

$$\begin{aligned}
\alpha_{3r} & = (-1)^r q^{\frac{9r^2-r}{2}} \frac{(e, q^3 u, q^3 v, q^5/u, q^5/v; q^3)_r}{(q^5/e, u, v, q^2/u, q^2/v; q^3)_r} e^r, \\
\alpha_{3r+1} & = (-1)^{r+1} q^{\frac{9r^2-r}{2}} \frac{(e, q^3 u, q^3 v, q^5/u, q^5/v; q^3)_r}{(q^5/e, u, v, q^2/u, q^2/v; q^3)_r} e^r + (-1)^r q^{\frac{9r^2+5r}{2}-2} \frac{(e/q^2, qu, qv, q^3/u, q^3/v; q^3)_{r+1}}{(q^3/e, u/q^2, v/q^2, 1/u, 1/v; q^3)_{r+1}} e^{r+1},
\end{aligned} \tag{2.21}$$

$$\alpha_{3r-1} = (-1)^r q^{\frac{9r^2-13r}{2}} \frac{(e/q^2, qu, qv, q^3/u, q^3/v; q^3)_r}{(q^3/e, u/q^2, v/q^2, 1/u, 1/v; q^3)_r} e^r,$$

$$\beta_n = K_3(n) \frac{(q^4/e; q^3)_n q^n}{(q^2; q)_{2n} (q^3/e; q)_n}.$$

(iii) the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = q^2$, where $\alpha_0 = \beta_0 = 1$, and

$$\alpha_{3r} = (-1)^r \frac{1 - q^{6r+2}}{1 - q^2} q^{\frac{9r^2-7r}{2}} \frac{(e, q^3u, q^3v, q^5/u, q^5/v; q^3)_r}{(q^5/e, u, v, q^2/u, q^2/v; q^3)_r} e^r, \quad (2.22)$$

$$\alpha_{3r+1} = (-1)^r \frac{1 - q^{6r+4}}{1 - q^2} q^{\frac{9r^2-r}{2}-3} \frac{(e/q^2, qu, qv, q^3/u, q^3/v; q^3)_{r+1}}{(q^3/e, u/q^2, v/q^2, 1/u, 1/v; q^3)_{r+1}} e^{r+1},$$

$$\alpha_{3r-1} = 0,$$

$$\beta_n = K_3(n) \frac{(q^4/e; q^3)_n}{(q^2; q)_{2n} (q^3/e; q)_n};$$

Proof. The proof parallels that of Theorem 2.5, except that after arriving at the identity

$$\sum \frac{(1 - q^{6r+2})(-1)^r q^{(9r^2-7r)/2} (e, q^3u, q^3v, q^5/u, q^5/v; q^3)_r}{(q^2; q)_{n+3r+1} (q; q)_{n-3r} e^r} \frac{(q^5/e, u, v, q^2/u, q^2/v; q^3)_r}{(q^2; q)_{2n} (q^3/e; q)_n} = \frac{K_3(n)(q^4/e; q^3)_n}{(q^2; q)_{2n} (q^3/e; q)_n}, \quad (2.23)$$

and then separating the sum into two sums ($r \geq 0$ and $r < 0$) as previously, we instead employ the identities

$$(1 - q^{6r+2})q^{\frac{9r^2-7r}{2}} = q^{\frac{9r^2-7r}{2}} (1 - q^{n+3r+2}) - q^{\frac{9r^2+5r}{2}+2} (1 - q^{n-3r}), \quad (2.24)$$

$$(1 - q^{-6r+2})q^{\frac{9r^2-7r}{2}} = q^{\frac{9r^2-7r}{2}} (1 - q^{n-3r+2}) - q^{\frac{9r^2-19r}{2}+2} (1 - q^{n+3r}),$$

to get (2.20). The result follows as in Theorem 2.5, except that it is necessary to re-index one of the four sums (by replacing r with $r + 1$).

For (2.21) we instead use the identities

$$(1 - q^{6r+2})q^{\frac{9r^2-7r}{2}} = q^{\frac{9r^2-r}{2}-n} ((1 - q^{n+3r+2}) - (1 - q^{n-3r})), \quad (2.25)$$

$$(1 - q^{-6r+2})q^{\frac{9r^2-7r}{2}} = q^{\frac{9r^2-13r}{2}-n} ((1 - q^{n-3r+2}) - (1 - q^{n+3r})).$$

The proof of (2.22) is like the proof of (2.16) in Theorem 2.5, except that after separating the sum at (2.23) into two sums, according to $r \geq 0$ or $r < 0$, and then replacing r with $-r$ for the sum with $r < 0$, we use the identities $(q^2; q)_{n+3r+1} (q; q)_{n-3r} = (1 - q^2)(q^3; q)_{n+3r} (q; q)_{n-3r}$ and $(q^2; q)_{n-3r+1} (q; q)_{n-3r} = (1 - q^2)(q; q)_{n-3r+2} (q^3; q)_{n+3r-2}$, and finally re-index in the latter case by replacing r with $r + 1$. \square

The nine Bailey pairs in the next corollary derive, respectively, from the pairs in Theorems 2.3 - 2.7, by letting $u, v \rightarrow \infty$ in each case.

Corollary 2.8. *The sequences (α_n, β_n) below are Bailey pairs with respect to the stated values of a , where*

$$\alpha_{3r} = \frac{1 - aq^{6r}}{1 - a} \frac{(a; q^3)_r}{(q^3; q^3)_r} q^{\frac{9r^2-3r}{2}} (-a)^r \quad (2.26)$$

$$\alpha_{3r\pm 1} = 0,$$

$$\beta_n = \frac{(a; q^3)_n}{(a; q)_{2n} (q; q)_n}, \quad \text{with respect to } a = a;$$

$$\alpha_{3r} = \frac{(aq; q^3)_r}{(q^3; q^3)_r} q^{\frac{9r^2-r}{2}} (-a)^r, \quad (2.27)$$

$$\alpha_{3r+1} = \frac{(aq; q^3)_r}{(q^3; q^3)_r} q^{\frac{9r^2+11r}{2}+1} (-a)^{r+1},$$

$$\alpha_{3r-1} = 0,$$

$$\beta_n = \frac{(aq; q^3)_n}{(aq; q)_{2n}(q; q)_n}, \quad \text{with respect to } a = a;$$

$$\begin{aligned} \alpha_{3r} &= \frac{(aq; q^3)_r}{(q^3; q^3)_r} q^{\frac{9r^2+5r}{2}} (-a)^r, \\ \alpha_{3r+1} &= -\frac{(aq; q^3)_r}{(q^3; q^3)_r} q^{\frac{9r^2+5r}{2}} (-a)^r, \\ \alpha_{3r-1} &= 0, \\ \beta_n &= \frac{(aq; q^3)_n q^n}{(aq; q)_{2n}(q; q)_n}, \quad \text{with respect to } a = a; \end{aligned} \tag{2.28}$$

$$\begin{aligned} \alpha_{3r} &= (-1)^r \left(\frac{q^{9r^2/2+r/2}(e; q^3)_r}{(q^4/e; q^3)_r e^r} + \frac{q^{9r^2/2+7r/2}(e/q; q^3)_r}{(q^3/e; q^3)_r e^r} \right), \\ \alpha_{3r+1} &= \frac{(-1)^{r+1} q^{9r^2/2+13r/2+1}(e; q^3)_r}{(q^4/e; q^3)_r e^r}, \\ \alpha_{3r-1} &= \frac{(-1)^{r+1} q^{9r^2/2-5r/2+1}(e/q; q^3)_r}{(q^3/e; q^3)_r e^r}, \\ \beta_n &= \frac{(q^2/e; q^3)_n}{(q; q)_{2n}(q^2/e; q)_n}, \quad \text{with respect to } a = 1; \end{aligned} \tag{2.29}$$

$$\begin{aligned} \alpha_{3r} &= (-1)^r \left(\frac{q^{9r^2/2+7r/2}(e; q^3)_r}{(q^4/e; q^3)_r e^r} + \frac{q^{9r^2/2+r/2}(e/q; q^3)_r}{(q^3/e; q^3)_r e^r} \right), \\ \alpha_{3r+1} &= \frac{(-1)^{r+1} q^{9r^2/2+7r/2}(e; q^3)_r}{(q^4/e; q^3)_r e^r}, \\ \alpha_{3r-1} &= \frac{(-1)^{r+1} q^{9r^2/2+r/2}(e/q; q^3)_r}{(q^3/e; q^3)_r e^r}, \\ \beta_n &= \frac{q^n (q^2/e; q^3)_n}{(q; q)_{2n}(q^2/e; q)_n}, \quad \text{with respect to } a = 1; \end{aligned} \tag{2.30}$$

$$\alpha_{3r} = (-1)^r \frac{1 - q^{6r+1}}{1 - q} q^{\frac{9r^2+r}{2}} \frac{(e; q^3)_r}{(q^4/e; q^3)_r e^r} \tag{2.31}$$

$$\alpha_{3r+1} = 0,$$

$$\alpha_{3r-1} = (-1)^{r+1} \frac{1 - q^{6r-1}}{1 - q} q^{\frac{9r^2-5r}{2}+1} \frac{(e/q; q^3)_r}{(q^3/e; q^3)_r e^r},$$

$$\beta_n = \frac{(q^2/e; q^3)_n}{(q; q)_{2n}(q^2/e; q)_n}, \quad \text{with respect to } a = q;$$

$$\begin{aligned} \alpha_{3r} &= \frac{(-1)^r q^{9r^2/2+5r/2}(e; q^3)_r}{(q^5/e; q^3)_r e^r}, \\ \alpha_{3r+1} &= \frac{(-1)^r q^{9r^2/2+11r/2+3}(e/q^2; q^3)_{r+1}}{(q^3/e; q^3)_{r+1} e^{r+1}} + \frac{(-1)^{r+1} q^{9r^2/2+17r/2+2}(e; q^3)_r}{(q^5/e; q^3)_r e^r}, \\ \alpha_{3r-1} &= \frac{(-1)^r q^{9r^2/2+5r/2}(e/q^2; q^3)_r}{(q^3/e; q^3)_r e^r}, \\ \beta_n &= \frac{(q^4/e; q^3)_n}{(q^2; q)_{2n}(q^3/e; q)_n}, \quad \text{with respect to } a = q; \end{aligned} \tag{2.32}$$

$$\begin{aligned}
\alpha_{3r} &= \frac{(-1)^r q^{9r^2/2+11r/2} (e; q^3)_r}{(q^5/e; q^3)_r e^r}, \\
\alpha_{3r+1} &= \frac{(-1)^r q^{9r^2/2+17r/2+4} (e/q^2; q^3)_{r+1}}{(q^3/e; q^3)_{r+1} e^{r+1}} + \frac{(-1)^{r+1} q^{9r^2/2+11r/2} (e; q^3)_r}{(q^5/e; q^3)_r e^r}, \\
\alpha_{3r-1} &= \frac{(-1)^r q^{9r^2/2-r/2} (e/q^2; q^3)_r}{(q^3/e; q^3)_r e^r}, \\
\beta_n &= \frac{q^n (q^4/e; q^3)_n}{(q^2; q)_{2n} (q^3/e; q)_n}, \quad \text{with respect to } a = q;
\end{aligned} \tag{2.33}$$

$$\begin{aligned}
\alpha_{3r} &= (-1)^r \frac{1 - q^{6r+2}}{1 - q^2} q^{\frac{9r^2+5r}{2}} \frac{(e; q^3)_r}{(q^5/e; q^3)_r e^r}, \\
\alpha_{3r+1} &= (-1)^r \frac{1 - q^{6r+4}}{1 - q^2} q^{\frac{9r^2+11r}{2}+3} \frac{(e/q^2; q^3)_{r+1}}{(q^3/e; q^3)_{r+1} e^{r+1}}, \\
\alpha_{3r-1} &= 0, \\
\beta_n &= \frac{(q^4/e; q^3)_n}{(q^2; q)_{2n} (q^3/e; q)_n} \quad \text{with respect to } a = q^2.
\end{aligned} \tag{2.34}$$

All eight pairs in Slater's **A** table ([16, page 463]) and all six on her **J** list ([17, pp. 148–149]) are derived from the five Bailey pairs (2.26), (2.29), (2.30), (2.32), (2.33) above, for particular values of e . Slater did not write down these general pairs above explicitly, but she could easily have derived them by the same methods she used to derive the special cases. None of Slater's pairs arise as special cases of (2.27), (2.28), (2.31) or (2.34), although the special case $e = -q^2$ of (2.31) was given in [9]. However, specializing the parameters give Bailey pairs which then give rise to some of the series-product identities on Slater's list, showing that different Bailey pairs may lead the same identity of Rogers-Ramanujan type.

Each of the nine Bailey pairs above also gives rise to a transformation between basic hypergeometric series, upon substituting the pair into (2.5). The pair at (2.26), for example, gives the following identity.

Corollary 2.9.

$$\sum_{n=0}^{\infty} \frac{(y, z; q)_n (a; q^3)_n}{(a; q)_{2n} (q; q)_n} \left(\frac{aq}{yz} \right)^n = \frac{(aq/y, aq/z; q)_{\infty}}{(aq, aq/yz; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(1 - aq^{6n})(y, z; q)_{3n} (a; q^3)_n a^{4n} q^{9n^2/2+3n/2}}{(1 - a)(aq/y, aq/z; q)_{3n} (q^3; q^3)_n} \left(\frac{-1}{yz} \right)^{3n}. \tag{2.35}$$

2.3 Mod 2 Bailey Pairs

We next consider how Slater produced the Bailey pairs in the **G**-, **C**- and **I** tables of [16, 17].

Theorem 2.10. *Let $K_4(n) := K(a, q^{-n}, q^{1-n}, d, a, u, v, q^2)$. Then*

(i) *the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = a$, where $\alpha_0 = \beta_0 = 1$ and*

$$\alpha_{2r} = \frac{1 - aq^{4r}}{1 - a} \frac{(a, d, q^2u, q^2v, aq^2/u, aq^2/v; q^2)_r q^{2r^2-4r} a^r}{(aq^2/d, u, v, a/u, a/v, q^2; q^2)_r d^r}, \tag{2.36}$$

$$\alpha_{2r-1} = 0,$$

$$\beta_n = K_4(n) \frac{(aq/d; q^2)_n}{(aq; q^2)_n (aq/d, q; q)_n}.$$

Let $K'_4(n) := K(aq, q^{-n}, q^{1-n}, d, aq, u, v, q^2)$. Then

(ii) *the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = a$, where $\alpha_0 = \beta_0 = 1$ and*

$$\alpha_{2r} = \frac{(aq, d, q^2u, q^2v, aq^3/u, aq^3/v; q^2)_r q^{2r^2-3r} a^r}{(aq^3/d, u, v, aq/u, aq/v, q^2; q^2)_r d^r}, \tag{2.37}$$

$$\alpha_{2r+1} = - \frac{(aq, d, q^2u, q^2v, aq^3/u, aq^3/v; q^2)_r q^{2r^2+r+1} a^{r+1}}{(aq^3/d, u, v, aq/u, aq/v, q^2; q^2)_r d^r},$$

$$\beta_n = K_4(n)' \frac{(aq^2/d; q^2)_n}{(aq^2; q^2)_n (aq^2/d, q; q)_n};$$

(iii) the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = a$, where $\alpha_0 = \beta_0 = 1$ and

$$\begin{aligned}\alpha_{2r} &= \frac{(aq, d, q^2u, q^2v, aq^3/u, aq^3/v; q^2)_r q^{2r^2-2r} a^r}{(aq^3/d, u, v, aq/u, aq/v, q^2; q^2)_r d^r}, \\ \alpha_{2r+1} &= -\frac{(aq, d, q^2u, q^2v, aq^3/u, aq^3/v; q^2)_r q^{2r^2-2r} a^r}{(aq^3/d, u, v, aq/u, aq/v, q^2; q^2)_r d^r}, \\ \beta_n &= K_4(n) \frac{(aq^2/d; q^2)_n q^n}{(aq^2; q^2)_n (aq^2/d, q; q)_n};\end{aligned}\tag{2.38}$$

Proof. The proof is quite similar to the proof of Theorem 2.3. As in the proof of that theorem, set $e = a$ in (1.3), so that all the terms of negative index vanish. Then replace q with q^2 , set $b = q^{-n}$ and $d = q^{1-n}$. After some simple manipulations, (1.3) becomes

$$\sum_{r=0}^{n/2} \frac{1 - aq^{4r}}{1 - a} \frac{(q^{-n}; q)_{2r}}{(aq^{n+1}; q)_{2r}} \frac{(a, d, q^2u, q^2v, aq^2/u, aq^2/v; q^2)_r}{(aq^2/du, v, a/u, a/v, q^2; q^2)_r} \left(\frac{aq^{2n-3}}{d}\right)^r = K_4(n) \frac{(aq/d; q^2)_n (aq; q)_n}{(aq; q^2)_n (aq/d; q)_n}.$$

Apply (2.2) to the $(q^{-n}; q)_{2r}$ factor, divide both sides by $(aq; q)_n (q; q)_n$ to get

$$\sum_{r=0}^{n/2} \frac{1 - aq^{4r}}{1 - a} \frac{q^{2r^2-4r}}{(aq; q)_{n+2r} (q; q)_{n-2r}} \frac{(a, d, q^2u, q^2v, aq^2/u, aq^2/v; q^2)_r}{(aq^2/du, v, a/u, a/v, q^2; q^2)_r} \left(\frac{a}{d}\right)^r = K_4(n) \frac{(aq/d; q^2)_n}{(aq; q^2)_n (aq/d, q; q)_n},\tag{2.39}$$

and the proof for the pair at (2.36) follows.

For (2.37) and (2.38), replace a with aq in (2.39), and use, respectively, the identities

$$\begin{aligned}(1 - aq^{4r+1})q^{2r^2-3r} &= q^{2r^2-3r}(1 - aq^{n+2r+1}) - aq^{2r^2+r+1}(1 - q^{n-2r}), \\ (1 - aq^{4r+1})q^{2r^2-3r} &= q^{2r^2-r} q^{-n} ((1 - aq^{n+2r+1}) - (1 - q^{n-2r})).\end{aligned}$$

□

Theorem 2.11. Let $K_5(n) := K(q, q^{-n}, q^{1-n}, d, e, u, v, q^2)$. Then

(i) the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = q$, where $\alpha_0 = \beta_0 = 1$ and

$$\begin{aligned}\alpha_{2r} &= \frac{1 - q^{4r+1}}{1 - q} \frac{(d, e, q^2u, q^2v, q^3/u, q^3/v; q^2)_r q^{2r^2-2r}}{(q^3/d, q^3/e, u, v, q/u, q/v; q^2)_r (de)^r}, \\ \alpha_{2r-1} &= -\frac{1 - q^{4r-1}}{1 - q} \frac{(d/q, e/q, qu, qv, q^2/u, q^2/v; q^2)_r q^{2r^2-4r+1}}{(q^2/d, q^2/e, u/q, v/q, 1/u, 1/v; q^2)_r (de)^r}, \\ \beta_n &= K_5(n) \frac{(q^3/de; q^2)_n}{(q^2; q^2)_n (q^2/d, q^2/e; q)_n};\end{aligned}\tag{2.40}$$

(ii) The pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = 1$, where $\alpha_0 = \beta_0 = 1$ and

$$\begin{aligned}\alpha_{2r} &= \frac{(d, e, q^2u, q^2v, q^3/u, q^3/v; q^2)_r q^{2r^2-2r}}{(q^3/d, q^3/e, u, v, q/u, q/v; q^2)_r (de)^r} + \frac{(d/q, e/q, qu, qv, q^2/u, q^2/v; q^2)_r q^{2r^2}}{(q^2/d, q^2/e, u/q, v/q, 1/u, 1/v; q^2)_r (de)^r}, \\ \alpha_{2r-1} &= -\frac{(d, e, q^2u, q^2v, q^3/u, q^3/v; q^2)_{r-1} q^{2r^2-2r+1}}{(q^3/d, q^3/e, u, v, q/u, q/v; q^2)_{r-1} (de)^{r-1}} - \frac{(d/q, e/q, qu, qv, q^2/u, q^2/v; q^2)_r q^{2r^2-4r+1}}{(q^2/d, q^2/e, u/q, v/q, 1/u, 1/v; q^2)_r (de)^r}, \\ \beta_n &= K_5(n) \frac{(q^3/de; q^2)_n}{(q^2; q^2)_n (q^2/d, q^2/e; q)_n};\end{aligned}\tag{2.41}$$

(iii) The pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = 1$, where $\alpha_0 = \beta_0 = 1$ and

$$\begin{aligned}\alpha_{2r} &= \frac{(d, e, q^2u, q^2v, q^3/u, q^3/v; q^2)_r q^{2r^2}}{(q^3/d, q^3/e, u, v, q/u, q/v; q^2)_r (de)^r} + \frac{(d/q, e/q, qu, qv, q^2/u, q^2/v; q^2)_r q^{2r^2-2r}}{(q^2/d, q^2/e, u/q, v/q, 1/u, 1/v; q^2)_r (de)^r}, \\ \alpha_{2r-1} &= -\frac{(d, e, q^2u, q^2v, q^3/u, q^3/v; q^2)_{r-1} q^{2r^2-4r+2}}{(q^3/d, q^3/e, u, v, q/u, q/v; q^2)_{r-1} (de)^{r-1}} - \frac{(d/q, e/q, qu, qv, q^2/u, q^2/v; q^2)_r q^{2r^2-2r}}{(q^2/d, q^2/e, u/q, v/q, 1/u, 1/v; q^2)_r (de)^r}, \\ \beta_n &= K_5(n) \frac{q^n (q^3/de; q^2)_n}{(q^2; q^2)_n (q^2/d, q^2/e; q)_n};\end{aligned}\tag{2.42}$$

Proof. The proof is very similar to the proof of Theorem 2.5. This time, replace q with q^2 in (1.3), and then set $a = q$, $b = q^{-n}$, $c = q^{1-n}$. The right side simplifies to give

$$K_5(n) \frac{(q, q^2; q)_n (q^3/de; q^2)_n}{(q^2; q^2)_n (q^2/d, q^2/e; q)_n}.$$

After some elementary manipulations, the series becomes

$$\sum_{r=-(n+1)/2}^{n/2} \frac{1 - q^{4r+1}}{1 - q} \frac{(q^{-n}; q)_{2r}}{(q^{n+2}; q)_{2r}} \frac{(d, e, q^2u, q^2v, q^3/u, q^3/v; q^2)_r}{(q^3/d, q^3/e, u, v, q/u, q/v; q^2)_r} \left(\frac{q^{2n-1}}{de} \right)^r.$$

We apply (2.2) to the $(q^{-n}; q)_{2r}$ factor above and rearrange terms to get

$$\sum_{r=-\frac{n+1}{2}}^{n/2} \frac{(1 - q^{4r+1})q^{2r^2-2r}}{(1 - q)(q^2; q)_{n+2r}(q; q)_{n-2r} d^r e^r} \frac{(d, e, q^2u, q^2v, q^3/u, q^3/v; q^2)_r}{(q^3/d, q^3/e, u, v, q/u, q/v; q^2)_r} = \frac{K_5(n)(q^3/de; q^2)_n}{(q^2; q^2)_n (q^2/d, q^2/e; q)_n}.$$

After applying (2.17) to the terms of negative index in the sum above, we get that

$$\begin{aligned} K_5(n) \frac{(q^3/de; q^2)_n}{(q^2; q^2)_n (q^2/d, q^2/e; q)_n} &= \sum_{r=0}^{n/2} \frac{(1 - q^{4r+1})q^{2r^2-2r}}{(1 - q)(q^2; q)_{n+2r}(q; q)_{n-2r} d^r e^r} \frac{(d, e, q^2u, q^2v, q^3/u, q^3/v; q^2)_r}{(q^3/d, q^3/e, u, v, q/u, q/v; q^2)_r} \\ &\quad - \sum_{r=1}^{\frac{n+1}{2}} \frac{(1 - q^{4r-1})q^{2r^2-4r+1}}{(1 - q)(q^2; q)_{n-2r}(q; q)_{n+2r} d^r e^r} \frac{(d/q, e/q, qu, qv, q^2/u, q^2/v; q^2)_r}{(q^2/d, q^2/e, u/q, v/q, 1/u, 1/v; q^2)_r}. \end{aligned} \quad (2.43)$$

The pair at (2.40) now follows after some simple manipulations, noting that

$$(q^2; q)_{n-2r}(q; q)_{n+2r} = (q; q)_{n-2r+1}(q^2; q)_{n+2r-1}.$$

The pair at (2.41) follows from (2.43), upon absorbing the $1 - q$ factor in the denominators there, employing the identities

$$\begin{aligned} (1 - q^{4r+1})q^{2r^2-2r} &= q^{2r^2-2r}(1 - q^{n+2r+1}) - q^{2r^2+2r+1}(1 - q^{n-2r}), \\ (1 - q^{4r-1})q^{2r^2-4r+1} &= q^{2r^2-4r+1}(1 - q^{n+2r}) - q^{2r^2}(1 - q^{n-2r+1}), \end{aligned} \quad (2.44)$$

in the way similar to the way that the pair of identities at (2.18) was used in the proof of Theorem 2.5, and finally re-indexing one of the resulting sums (by replacing r with $r - 1$).

The proof for the pair at (2.42) is similar to the proof for the pair at (2.41), except we employ the identities

$$\begin{aligned} (1 - q^{4r+1})q^{2r^2-2r} &= q^{2r^2-n}((1 - q^{n+2r+1}) - (1 - q^{n-2r})), \\ (1 - q^{4r-1})q^{2r^2-4r+1} &= q^{2r^2-2r-n}((1 - q^{n+2r}) - (1 - q^{n-2r+1})). \end{aligned} \quad (2.45)$$

□

As with $K_0 - K_3$, K_4 and K_5 are in general quite complicated, but simplify considerably for particular values of the parameters, leading (as was the case in Corollary 2.2) to transformations of basic hypergeometric series. We give one example.

Corollary 2.12. *If $a, y, z, q \in \mathbb{C}$ such that $|q| < 1$ and none of the denominators below vanish, then*

$$\sum_{n=0}^{\infty} \frac{(y, z; q)_n (-1/q; q^2)_n}{(q^2; q^2)_n (aq; q^2)_n} \left(\frac{aq^2}{yz} \right)^n = \frac{(aq/y, aq/z; q)_{\infty}}{(aq, aq/yz; q)_{\infty}} \sum_{n=0}^{\infty} \frac{1 - aq^{4n}}{1 - a} \frac{(y, z; q)_{2n} (a^2, -aq^4; q^4)_n}{(aq/y, aq/z; q)_{2n} (-a, q^4; q^4)_n} \left(\frac{-a^2}{y^2 z^2} \right)^n q^{2n^2}. \quad (2.46)$$

Proof. In the Bailey pair at (2.36), set $d = -a/q^2$, $u = -q^2$ and $v = i\sqrt{a}$, so that $K_4(n)$ takes the value

$$K \left(a, q^{-n}, q^{1-n}, \frac{-a}{q^2}, a, -q^2, i\sqrt{a}, q^2 \right) = \frac{q^{n-1}(q+1)(q^{n+1}+1)(q^{n+2}+1)}{(q^2+1)(q^{2n-1}+1)(q^{2n+1}+1)}.$$

Substitute the resulting Bailey pair into (2.5), and the result follows after some elementary q -product manipulations. □

The Bailey pairs in Slater's **G**-, **C**- and **I** tables, as well as pairs **E(1)**, **E(2)**, **E(4)** and **E(5)** (see [16, pages 469 and 470]), are derived from the next six Bailey pairs. These, in turn, are derived from the pairs in Theorems 2.10 and 2.11, by letting $u, v \rightarrow \infty$.

Corollary 2.13. *The sequences (α_n, β_n) below are Bailey pairs with respect to the stated values of a , where $\alpha_0 = \beta_0 = 1$ and*

$$\alpha_{2r} = \frac{1 - aq^{4r}}{1 - a} \frac{(a, d; q^2)_r a^r q^{2r^2}}{(aq^2/d, q^2; q^2)_r d^r}, \quad (2.47)$$

$$\alpha_{2r-1} = 0,$$

$$\beta_n = \frac{(aq/d; q^2)_n}{(aq; q^2)_n (aq/d, q; q)_n}, \quad \text{with respect to } a = a;$$

$$\alpha_{2r} = \frac{(aq, d; q^2)_r q^{2r^2+r} a^r}{(aq^3/d, q^2; q^2)_r d^r}, \quad (2.48)$$

$$\alpha_{2r+1} = -\frac{(aq, d; q^2)_r q^{2r^2+5r+1} a^{r+1}}{(aq^3/d, q^2; q^2)_r d^r},$$

$$\beta_n = \frac{(aq^2/d; q^2)_n}{(aq^2; q^2)_n (aq^2/d, q; q)_n} \quad \text{with respect to } a = a;$$

$$\alpha_{2r} = \frac{(aq, d; q^2)_r q^{2r^2+3r} a^r}{(aq^3/d, q^2; q^2)_r d^r}, \quad (2.49)$$

$$\alpha_{2r+1} = -\frac{(aq, d; q^2)_r q^{2r^2+3r} a^r}{(aq^3/d, q^2; q^2)_r d^r},$$

$$\beta_n = \frac{(aq^2/d; q^2)_n q^n}{(aq^2; q^2)_n (aq^2/d, q; q)_n}, \quad \text{with respect to } a = a;$$

$$\alpha_{2r} = \frac{(1 - q^{4r+1})q^{2r^2+2r} (d; q^2)_r (e; q^2)_r}{(1 - q)(q^3/d; q^2)_r (q^3/e; q^2)_r d^r e^r}, \quad (2.50)$$

$$\alpha_{2r-1} = -\frac{(1 - q^{4r-1})q^{2r^2+1} (d/q; q^2)_r (e/q; q^2)_r}{(1 - q)(q^2/d; q^2)_r (q^2/e; q^2)_r d^r e^r},$$

$$\beta_n = \frac{(q^3/de; q^2)_n}{(q^2; q^2)_n (q^2/d; q)_n (q^2/e; q)_n}, \quad \text{with respect to } a = q;$$

$$\alpha_{2r} = \frac{q^{2r^2+4r} (d/q; q^2)_r (e/q; q^2)_r}{(q^2/d; q^2)_r (q^2/e; q^2)_r d^r e^r} + \frac{q^{2r^2+2r} (d; q^2)_r (e; q^2)_r}{(q^3/d; q^2)_r (q^3/e; q^2)_r d^r e^r}, \quad (2.51)$$

$$\alpha_{2r+1} = -\frac{q^{2r^2+4r+3} (d/q; q^2)_{r+1} (e/q; q^2)_{r+1}}{(q^2/d; q^2)_{r+1} (q^2/e; q^2)_{r+1} d^{r+1} e^{r+1}} - \frac{q^{2r^2+6r+1} (d; q^2)_r (e; q^2)_r}{(q^3/d; q^2)_r (q^3/e; q^2)_r d^r e^r},$$

$$\beta_n = \frac{(q^3/de; q^2)_n}{(q^2; q^2)_n (q^2/d; q)_n (q^2/e; q)_n}, \quad \text{with respect to } a = 1;$$

$$\alpha_{2r} = \frac{q^{2r^2+2r} (d/q; q^2)_r (e/q; q^2)_r}{(q^2/d; q^2)_r (q^2/e; q^2)_r d^r e^r} + \frac{q^{2r^2+4r} (d; q^2)_r (e; q^2)_r}{(q^3/d; q^2)_r (q^3/e; q^2)_r d^r e^r}, \quad (2.52)$$

$$\alpha_{2r+1} = -\frac{q^{2r^2+6r+4} (d/q; q^2)_{r+1} (e/q; q^2)_{r+1}}{(q^2/d; q^2)_{r+1} (q^2/e; q^2)_{r+1} d^{r+1} e^{r+1}} - \frac{q^{2r^2+4r} (d; q^2)_r (e; q^2)_r}{(q^3/d; q^2)_r (q^3/e; q^2)_r d^r e^r},$$

$$\beta_n = \frac{q^n (q^3/de; q^2)_n}{(q^2; q^2)_n (q^2/d; q)_n (q^2/e; q)_n}, \quad \text{with respect to } a = 1.$$

2.4 Mod 4 Bailey Pairs

We continue to follow in Slater's path [16], next considering how she produced the Bailey pairs in her \mathbf{K} table.

Theorem 2.14. *Let $K_6(n) := K(q, q^{-n}, q^{1-n}, q^{2-n}, q^{3-n}, u, v, q^4)$ for $n \geq 3$, and $K_6(0) = K_6(1) = K_6(2) = 1$. Then (i) the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = 1$, where $\alpha_0 = \beta_0 = 1$, $\alpha_{4r+2} = 0$, and*

$$\begin{aligned}\alpha_{4r} &= \frac{(q^4 u, q^4 v, q^5/u, q^5/v; q^4)_r}{(u, v, q/u, q/v; q^4)_r} q^{8r^2-10r} + \frac{(q^3 u, q^3 v, q^4/u, q^4/v; q^4)_r}{(u/q, v/q, 1/u, 1/v; q^4)_r} q^{8r^2-6r} \\ \alpha_{4r+1} &= -\frac{(q^4 u, q^4 v, q^5/u, q^5/v; q^4)_r}{(u, v, q/u, q/v; q^4)_r} q^{8r^2-2r+1} \\ \alpha_{4r-1} &= -\frac{(q^3 u, q^3 v, q^4/u, q^4/v; q^4)_r}{(u/q, v/q, 1/u, 1/v; q^4)_r} q^{8r^2-14r+1} \\ \beta_n &= K_6(n) \frac{(-q^2; q^2)_{n-1}}{(q; q)_{2n}};\end{aligned}\tag{2.53}$$

(ii) the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = 1$, where $\alpha_0 = \beta_0 = 1$, $\alpha_{4r+2} = 0$, and

$$\begin{aligned}\alpha_{4r} &= \frac{(q^4 u, q^4 v, q^5/u, q^5/v; q^4)_r}{(u, v, q/u, q/v; q^4)_r} q^{8r^2-6r} + \frac{(q^3 u, q^3 v, q^4/u, q^4/v; q^4)_r}{(u/q, v/q, 1/u, 1/v; q^4)_r} q^{8r^2-10r} \\ \alpha_{4r+1} &= -\frac{(q^4 u, q^4 v, q^5/u, q^5/v; q^4)_r}{(u, v, q/u, q/v; q^4)_r} q^{8r^2-6r} \\ \alpha_{4r-1} &= -\frac{(q^3 u, q^3 v, q^4/u, q^4/v; q^4)_r}{(u/q, v/q, 1/u, 1/v; q^4)_r} q^{8r^2-10r} \\ \beta_n &= K_6(n) \frac{q^n (-q^2; q^2)_{n-1}}{(q; q)_{2n}}.\end{aligned}\tag{2.54}$$

Proof. The proof is initially similar to the proof of Theorem 2.3, except we replace q with q^4 and set $b = q^{-n}$, $c = q^{1-n}$, $d = q^{2-n}$ and $e = q^{3-n}$. Instead of (2.13), we arrive at

$$\begin{aligned}\sum_{r \in \mathbb{Z}} \frac{1 - aq^{8r}}{1 - a} \frac{a^{2r} q^{8r^2-12r}}{(aq; q)_{n+4r} (q; q)_{n-4r}} \frac{(q^4 u, q^4 v, aq^4/u, aq^4/v; q^4)_r}{(u, v, a/u, a/v; q^4)_r} \\ = K(a, q^{-n}, q^{1-n}, q^{2-n}, q^{3-n}, u, v, q^4) \frac{(q^4/a, a/q, a, aq; q^4)_\infty}{(q, q^2, q^3, a^2/q^2; q^4)_\infty} \frac{(-a/q; q^2)_n}{(a; q)_{2n}}.\end{aligned}\tag{2.55}$$

Next, separate the sum into terms of positive and negative index, re-index the latter sum by replacing r with $-r$ (and also using (2.17)), to get

$$\begin{aligned}\sum_{r \geq 0} \frac{1 - aq^{8r}}{1 - a} \frac{a^{2r} q^{8r^2-12r}}{(aq; q)_{n+4r} (q; q)_{n-4r}} \frac{(q^4 u, q^4 v, aq^4/u, aq^4/v; q^4)_r}{(u, v, a/u, a/v; q^4)_r} \\ + \sum_{r \geq 1} \frac{1 - aq^{-8r}}{1 - a} \frac{a^{-2r} q^{8r^2-4r}}{(aq; q)_{n-4r} (q; q)_{n+4r}} \frac{(q^4/u, q^4/v, uq^4/a, vq^4/a; q^4)_r}{(1/u, 1/v, u/a, v/a; q^4)_r} \\ = K(a, q^{-n}, q^{1-n}, q^{2-n}, q^{3-n}, u, v, q^4) \frac{(q^4/a, a/q, a, aq; q^4)_\infty}{(q, q^2, q^3, a^2/q^2; q^4)_\infty} \frac{(-a/q; q^2)_n}{(a; q)_{2n}}.\end{aligned}\tag{2.56}$$

The proof for the Bailey pair at (2.53) then follows, upon setting $a = q$, simplifying the product side, and using the identities

$$\begin{aligned}(1 - q^{8r+1})q^{8r^2-10r} &= q^{8r^2-10r}(1 - q^{n+4r+1}) - q^{8r^2-2r+1}(1 - q^{n-4r}), \\ (1 - q^{-8r+1})q^{8r^2-6r} &= q^{8r^2-6r}(1 - q^{n-4r+1}) - q^{8r^2-14r+1}(1 - q^{n+4r}).\end{aligned}$$

The proof for the Bailey pair at (2.54) is similar, except we use the identities

$$\begin{aligned}(1 - q^{8r+1})q^{8r^2-10r} &= q^{-n} q^{8r^2-6r} ((1 - q^{n+4r+1}) - (1 - q^{n-4r})), \\ (1 - q^{-8r+1})q^{8r^2-6r} &= q^{-n} q^{8r^2-10r} ((1 - q^{n-4r+1}) - (1 - q^{n+4r})).\end{aligned}$$

□

Theorem 2.15. Let $K_7(n) := K(q^2, q^{-n}, q^{1-n}, q^{2-n}, q^{3-n}, u, v, q^4)$ for $n \geq 0$. Then (i) the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = q$, where $\alpha_0 = \beta_0 = 1$, and

$$\begin{aligned}\alpha_{4r} &= \frac{(q^4 u, q^4 v, q^6/u, q^6/v; q^4)_r}{(u, v, q^2/u, q^2/v; q^4)_r} q^{8r^2-8r} \\ \alpha_{4r+1} &= -\frac{(q^4 u, q^4 v, q^6/u, q^6/v; q^4)_r}{(u, v, q^2/u, q^2/v; q^4)_r} q^{8r^2+2} \\ \alpha_{4r-1} &= \frac{(q^2 u, q^2 v, q^4/u, q^4/v; q^4)_r}{(u/q^2, v/q^2, 1/u, 1/v; q^4)_r} q^{8r^2-8r} \\ \alpha_{4r-2} &= -\frac{(q^2 u, q^2 v, q^4/u, q^4/v; q^4)_r}{(u/q^2, v/q^2, 1/u, 1/v; q^4)_r} q^{8r^2-16r+2} \\ \beta_n &= K_7(n) \frac{(-q; q^2)_n}{(q^2; q)_{2n}};\end{aligned}\tag{2.57}$$

(ii) the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = q$, where $\alpha_0 = \beta_0 = 1$, and

$$\begin{aligned}\alpha_{4r} &= \frac{(q^4 u, q^4 v, q^6/u, q^6/v; q^4)_r}{(u, v, q^2/u, q^2/v; q^4)_r} q^{8r^2-4r} \\ \alpha_{4r+1} &= -\frac{(q^4 u, q^4 v, q^6/u, q^6/v; q^4)_r}{(u, v, q^2/u, q^2/v; q^4)_r} q^{8r^2-4r} \\ \alpha_{4r-1} &= \frac{(q^2 u, q^2 v, q^4/u, q^4/v; q^4)_r}{(u/q^2, v/q^2, 1/u, 1/v; q^4)_r} q^{8r^2-12r} \\ \alpha_{4r-2} &= -\frac{(q^2 u, q^2 v, q^4/u, q^4/v; q^4)_r}{(u/q^2, v/q^2, 1/u, 1/v; q^4)_r} q^{8r^2-12r} \\ \beta_n &= K_7(n) \frac{q^n (-q; q^2)_n}{(q^2; q)_{2n}}.\end{aligned}\tag{2.58}$$

Proof. Set $a = q^2$ in (2.56), simplify the product side and absorb the $1 - q^2$ terms on the sum side into what were previously the $(aq; q)_{n+4r}$ and $(aq; q)_{n-4r}$ factors. The pair at (2.57) follows after using the identities

$$\begin{aligned}(1 - q^{8r+2})q^{8r^2-8r} &= q^{8r^2-8r} (1 - q^{n+4r+2}) - q^{8r^2+2} (1 - q^{n-4r}), \\ (1 - q^{-8r+2})q^{8r^2-8r} &= q^{8r^2-8r} (1 - q^{n-4r+2}) - q^{8r^2-16r+2} (1 - q^{n+4r}).\end{aligned}$$

The pair at (2.58) follows similarly, after employing the identities

$$\begin{aligned}(1 - q^{8r+2})q^{8r^2-8r} &= q^{-n} q^{8r^2-4r} ((1 - q^{n+4r+2}) - (1 - q^{n-4r})), \\ (1 - q^{-8r+2})q^{8r^2-8r} &= q^{-n} q^{8r^2-12r} ((1 - q^{n-4r+2}) - (1 - q^{n+4r})).\end{aligned}$$

□

Theorem 2.16. Let $K_8(0) = 1$,

$$K_8(1) = \frac{(q-u)(q^2-u)(q-v)(q^2-v)}{(q^3-u)(u-1)(q^3-v)(v-1)} - \frac{(q-1)^4(q+1)^2(q^2+q+1)uv}{q(q^3-u)(u-1)(q^3-v)(v-1)},$$

and $K_8(n) := K(q^3, q^{-n}, q^{1-n}, q^{2-n}, q^{3-n}, u, v, q^4)$ for $n \geq 2$. Then

(i) the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = q^2$, where $\alpha_0 = \beta_0 = 1$, and

$$\begin{aligned}\alpha_{4r} &= \frac{(q^4 u, q^4 v, q^7/u, q^7/v; q^4)_r}{(u, v, q^3/u, q^3/v; q^4)_r} q^{8r^2-6r} \\ \alpha_{4r+1} &= -\frac{(q^4 u, q^4 v, q^7/u, q^7/v; q^4)_r}{(u, v, q^3/u, q^3/v; q^4)_r} q^{8r^2+2r+3} - \frac{(q^4/u, q^4/v, qu, qv; q^4)_{r+1}}{(1/u, 1/v, u/q^3, v/q^3; q^4)_{r+1}} q^{8r^2-2r-7} \\ \alpha_{4r-1} &= 0\end{aligned}\tag{2.59}$$

$$\alpha_{4r-2} = \frac{(q^4/u, q^4/v, qu, qv; q^4)_r}{(1/u, 1/v, u/q^3, v/q^3; q^4)_r} q^{8r^2-10r}$$

$$\beta_n = K_8(n) \frac{(-q^2; q^2)_n}{(q^3; q)_{2n}};$$

(ii) the pair of sequences (α_n, β_n) is a Bailey pair with respect to $a = q^2$, where $\alpha_0 = \beta_0 = 1$, and

$$\alpha_{4r} = \frac{(q^4u, q^4v, q^7/u, q^7/v; q^4)_r}{(u, v, q^3/u, q^3/v; q^4)_r} q^{8r^2-2r} \tag{2.60}$$

$$\alpha_{4r+1} = -\frac{(q^4u, q^4v, q^7/u, q^7/v; q^4)_r}{(u, v, q^3/u, q^3/v; q^4)_r} q^{8r^2-2r} - \frac{(q^4/u, q^4/v, qu, qv; q^4)_{r+1}}{(1/u, 1/v, u/q^3, v/q^3; q^4)_{r+1}} q^{8r^2+2r-6}$$

$$\alpha_{4r-1} = 0$$

$$\alpha_{4r-2} = \frac{(q^4/u, q^4/v, qu, qv; q^4)_r}{(1/u, 1/v, u/q^3, v/q^3; q^4)_r} q^{8r^2-14r}$$

$$\beta_n = K_8(n) q^n \frac{(-q^2; q^2)_n}{(q^3; q)_{2n}};$$

Remark 2.17. The reason $K_8(1)$ does not fit into the formula for $K_8(n)$ for $n > 1$ is that it is necessary to be careful with the order in which (after replacing q with q^4) we set $a = q^3$, $b = 1/q$, $c = 1$, $d = q$ and $e = q^2$, in order to get the series at (1.3) to converge. It is easy to see that making those substitutions simultaneously gives $a^2/q^4bcde = 1$, contradicting the requirement that $|a^2/q^4bcde| < 1$ in the series. One way around this is to first set $c = 1$ and $b = a/q^4$, causing the series to terminate above and below, after which the other replacements can be made.

Note also that, with the stated choices for the parameters, the $a^2 - bcdeq^4$ factor in the denominator of the expression for K at (1.2) vanishes, and in fact these choices also cause the numerator of K to vanish. It is cancelling these “zero factors” that make the expression for $K_8(1)$ different. The order of substitutions described above causes $K(q^3, q^{-1}, 1, q, q^2, u, v, q^4)$ to have the value assigned to $K_8(1)$, whereas for $n > 1$, $K(q^3, q^{-n}, q^{1-n}, q^{2-n}, q^{3-n}, u, v, q^4)$ is independent of the order of substitutions.

We also note that a similar situation occurs in some of the other theorems above.

Proof. The proof is similar to the proof of Theorem 2.15, except we set $a = q^3$ in (2.56). The pair at (2.59) follows after using the identities

$$(1 - q^{8r+3})q^{8r^2-6r} = q^{8r^2-6r}(1 - q^{n+4r+3}) - q^{8r^2+2r+3}(1 - q^{n-4r}),$$

$$(1 - q^{-8r+3})q^{8r^2-10r} = q^{8r^2-10r}(1 - q^{n-4r+3}) - q^{8r^2-18r+3}(1 - q^{n+4r}),$$

and then replacing r with $r+1$ in the sum corresponding to the second term in the second identity above (so that this sum implicitly defines part of α_{4r+1} instead of α_{4r-3}). The pair at (2.60) follows similarly, after using the identities

$$(1 - q^{8r+3})q^{8r^2-6r} = q^{-n}q^{8r^2-2r}((1 - q^{n+4r+3}) - (1 - q^{n-4r})),$$

$$(1 - q^{-8r+3})q^{8r^2-10r} = q^{-n}q^{8r^2-14r}((1 - q^{n-4r+3}) - (1 - q^{n+4r})).$$

□

The six Bailey pairs in Slater’s **K** table [16, page 471] follow from the three theorems in this section, upon letting $u, v \rightarrow \infty$.

Corollary 2.18. *The following pairs of sequences (α_n, β_n) , with $\alpha_0 = \beta_0 = 1$, are Bailey pairs with respect to the stated value of a .*

$$\alpha_{4r} = q^{8r^2-2r} + q^{8r^2+2r} \tag{2.61}$$

$$\alpha_{4r+1} = -q^{8r^2+6r+1}$$

$$\alpha_{4r-1} = -q^{8r^2-6r+1}$$

$$\alpha_{4r-2} = 0$$

$$\beta_n = \frac{(-q^2; q^2)_{n-1}}{(q; q)_{2n}} \text{ with respect to } a = 1;$$

$$\alpha_{4r} = q^{8r^2+2r} + q^{8r^2-2r} \tag{2.62}$$

$$\alpha_{4r+1} = -q^{8r^2+2r}$$

$$\alpha_{4r-1} = -q^{8r^2-2r}$$

$$\alpha_{4r-2} = 0$$

$$\beta_n = \frac{q^n (-q^2; q^2)_{n-1}}{(q; q)_{2n}} \text{ with respect to } a = 1;$$

$$\alpha_{4r} = q^{8r^2} \tag{2.63}$$

$$\alpha_{4r+1} = -q^{8r^2+8r+2}$$

$$\alpha_{4r-1} = q^{8r^2}$$

$$\alpha_{4r-2} = -q^{8r^2-8r+2}$$

$$\beta_n = \frac{(-q; q^2)_n}{(q^2; q)_{2n}} \text{ with respect to } a = q;$$

$$\alpha_{4r} = q^{8r^2+4r} \tag{2.64}$$

$$\alpha_{4r+1} = -q^{8r^2+4r}$$

$$\alpha_{4r-1} = q^{8r^2-4r}$$

$$\alpha_{4r-2} = -q^{8r^2-4r}$$

$$\beta_n = \frac{q^n (-q; q^2)_n}{(q^2; q)_{2n}} \text{ with respect to } a = q;$$

$$\alpha_{4r} = q^{8r^2+2r} \tag{2.65}$$

$$\alpha_{4r+1} = -q^{8r^2+10r+3} - q^{8r^2+6r+1}$$

$$\alpha_{4r-1} = 0$$

$$\alpha_{4r-2} = q^{8r^2-2r}$$

$$\beta_n = \frac{(-q^2; q^2)_n}{(q^3; q)_{2n}} \text{ with respect to } a = q^2;$$

$$\alpha_{4r} = q^{8r^2+6r} \tag{2.66}$$

$$\alpha_{4r+1} = -q^{8r^2+6r} - q^{8r^2+10r+2}$$

$$\alpha_{4r-1} = 0$$

$$\alpha_{4r-2} = q^{8r^2-6r}$$

$$\beta_n = q^n \frac{(-q^2; q^2)_n}{(q^3; q)_{2n}} \text{ with respect to } a = q^2.$$

3 Identities of the Rogers-Ramanujan-Slater type and False Theta Series Identities

We use the Bailey pairs found in the previous section to derive several new series-product identities, and also to give new proofs of some general identities due to Ramanujan, Andrews and others. We also remark that we sometimes use a more general case of the ${}_6\psi_6$ than the Jacobi Triple Product Identity, in contrast to how Slater derived her products.

3.1 General identities containing one or more free parameters

We first give new proofs for two quite general identities of Ramanujan ([12, page 33], see also R1 and R2 on page 8 of [10]). One reason these two identities are of interest is that each leads to infinitely many identities of Rogers-Ramanujan type (set $z = \pm q^{a/b}$, where a and $b > 0$ are integers, and then replace q with q^b).

Theorem 3.1. For $|q| < 1$ and $z \in \mathbb{C} \setminus \{0\}$,

$$\sum_{n=0}^{\infty} \frac{(q/z; q)_n (z; q)_n q^{n^2}}{(q; q)_{2n}} = \frac{(qz, q^2/z, q^3; q^3)_{\infty}}{(q; q)_{\infty}}. \quad (3.1)$$

Proof. Let $e \rightarrow 0$ in the Bailey pair at (2.31), and then insert the resulting pair into (2.5), after setting $y = q/z$. This leads to the identities

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(q/z; q)_n (z; q)_n q^{n^2}}{(q; q)_{2n}} &= \frac{(qz, \frac{q^2}{z}; q)_{\infty}}{(q^2, q; q)_{\infty}} \left[\sum_{r=0}^{\infty} \frac{1 - q^{6r+1}}{1 - q} \frac{(q/z, z; q)_{3r} q^{3r^2+r}}{(qz, \frac{q^2}{z}; q)_{3r}} - \sum_{r=1}^{\infty} \frac{1 - q^{6r-1}}{1 - q} \frac{(q/z, z; q)_{3r-1} q^{3r^2-r}}{(qz, q^2/z; q)_{3r-1}} \right] \\ &= \frac{(qz, \frac{q^2}{z}; q)_{\infty}}{(q^2, q; q)_{\infty}} \left[\sum_{r=0}^{\infty} \frac{1 - q^{6r+1}}{1 - q} \frac{(q/z, z; q^3)_r q^{3r^2+r}}{(q^3 z, \frac{q^4}{z}; q^3)_r} + \sum_{r=1}^{\infty} \frac{1 - q^{6r-1}}{1 - 1/q} \frac{(1/z, z/q; q^3)_r q^{3r^2-r}}{(q^2 z, q^3/z; q^3)_r} \right] \\ &= \frac{(qz, \frac{q^2}{z}; q)_{\infty}}{(q^2, q; q)_{\infty}} \frac{(q^4, q^3, q^3, q^2; q^3)_{\infty}}{(q^4/z, zq^3, q^3/z, q^2z; q^3)_{\infty}}. \end{aligned}$$

The second equality above follows after some elementary q -product manipulations and the last equality follows from the fact that the two series in the previous equality combine to give a special case of Bailey's ${}_6\psi_6$ summation at (1.1) (replace q with q^3 , let $d, e \rightarrow \infty$, set $a = q$, $b = q/z$ and $c = z$). The result now follows after some further elementary manipulations. \square

Theorem 3.2. For $|q| < 1$ and $z \in \mathbb{C} \setminus \{0\}$,

$$\sum_{n=0}^{\infty} \frac{(q^2/z; q^2)_n (z; q^2)_n q^{n^2}}{(q; q^2)_n (q^4; q^4)_n} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} (qz, q^3/z, q^4; q^4)_{\infty}. \quad (3.2)$$

Proof. The proof is similar to that of the theorem above. Let $e \rightarrow 0$ and set $d = q^{3/2}$ in the Bailey pair at (2.50), and then insert the resulting pair into (2.5), after setting $y = q/z$. This leads to the identities

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(q/z; q)_n (z; q)_n q^{n^2/2}}{(q^{1/2}; q)_n (q^2; q^2)_n} &= \frac{(qz, \frac{q^2}{z}; q)_{\infty}}{(q^2, q; q)_{\infty}} \left[\sum_{r=0}^{\infty} \frac{1 - q^{4r+1}}{1 - q} \frac{(q/z, z; q)_{2r} q^{r^2+r/2}}{(qz, \frac{q^2}{z}; q)_{2r} (-1)^r} - \sum_{r=1}^{\infty} \frac{1 - q^{4r-1}}{1 - q} \frac{(q/z, z; q)_{2r-1} q^{r^2-r/2} (-1)^r}{(qz, q^2/z; q)_{2r-1}} \right] \\ &= \frac{(qz, \frac{q^2}{z}; q)_{\infty}}{(q^2, q; q)_{\infty}} \left[\sum_{r=0}^{\infty} \frac{1 - q^{4r+1}}{1 - q} \frac{(q/z, z; q^2)_r (-1)^r q^{r^2+r/2}}{(q^2 z, \frac{q^3}{z}; q^2)_r} + \sum_{r=1}^{\infty} \frac{1 - q^{4r-1}}{1 - 1/q} \frac{(1/z, z/q; q^2)_r (-1)^r q^{r^2-r/2}}{(qz, q^2/z; q^2)_r} \right] \\ &= \frac{(qz, \frac{q^2}{z}; q)_{\infty}}{(q^2, q; q)_{\infty}} \frac{(q^3, q^2, zq^{1/2}, q^{3/2}/z, q^2, q; q^2)_{\infty}}{(q^3/z, zq^2, q^{3/2}, q^2/z, qz, q^{1/2}; q^2)_{\infty}}. \end{aligned}$$

The second equality above once again follows after some elementary q -product manipulations and once again the two series on the right side in the second equality combine to give a special case of Bailey's ${}_6\psi_6$ summation at (1.1) (replace q with q^2 , let $e \rightarrow \infty$, set $a = q$, $b = q/z$, $c = z$ and $d = q^{3/2}$). The result now follows after replacing q with q^2 , followed by some further elementary manipulations. \square

Different proofs of the results in the two theorems above were also given in [4] (Entries 5.3.1 and 5.3.5) and in [11].

We now prove a new general series-product identity, one which may be regarded as a partner to Ramanujan's result in Theorem 3.1.

Theorem 3.3. For $|q| < 1$ and $z \in \mathbb{C} \setminus \{0\}$,

$$\sum_{n=0}^{\infty} \frac{(q/z; q)_{n+1} (z; q)_n q^{n^2+n}}{(q; q)_{2n+1}} = \frac{(q^2 z, q/z, q^3; q^3)_{\infty}}{(q; q)_{\infty}}. \quad (3.3)$$

Proof. Let $e \rightarrow 0$ in the Bailey pair at (2.34), and then insert the resulting pair into (2.5), after setting $y = q^2/z$. This leads to the identities

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\left(\frac{q^2}{z}; q\right)_n (z; q)_n q^{n^2+n}}{(q^2; q)_{2n}} \\ &= \frac{\left(qz, \frac{q^3}{z}; q\right)_{\infty}}{(q^3, q; q)_{\infty}} \left[\sum_{r=0}^{\infty} \frac{1 - q^{6r+2}}{1 - q^2} \frac{\left(\frac{q^2}{z}, z; q\right)_{3r} q^{3r^2+2r}}{\left(qz, \frac{q^3}{z}; q\right)_{3r}} - \sum_{r=0}^{\infty} \frac{1 - q^{6r+4}}{1 - q^2} \frac{(q^2/z, z; q)_{3r+1} q^{3r^2+4r+1}}{(qz, q^3/z; q)_{3r+1}} \right] \\ &= \frac{\left(qz, \frac{q^3}{z}; q\right)_{\infty}}{(q^3, q; q)_{\infty}} \left[\sum_{r=0}^{\infty} \frac{1 - q^{6r+2}}{1 - q^2} \frac{\left(\frac{q^2}{z}, z; q^3\right)_r q^{3r^2+2r}}{\left(q^3 z, \frac{q^3}{z}; q^3\right)_r} + \sum_{r=1}^{\infty} \frac{1 - q^{6r-2}}{1 - 1/q^2} \frac{(1/z, z/q^2; q^3)_r q^{3r^2-2r}}{(qz, q^3/z; q^3)_r} \right] \\ &= \frac{\left(qz, \frac{q^3}{z}; q\right)_{\infty}}{(q^3, q; q)_{\infty}} \frac{(q^5, q^3, q^3, q; q^3)_{\infty}}{(q^5/z, zq^3, q^3/z, qz; q^3)_{\infty}}. \end{aligned}$$

The second equality above follows after some elementary q -product manipulations and re-indexing the second series (replacing r with $r - 1$). The last equality follows from the fact that the two series in the previous equality combine to give another special case of Bailey's ${}_6\psi_6$ summation (1.1) (replace q with q^3 , let $d, e \rightarrow \infty$, set $a = q^2$, $b = q^2/z$ and $c = z$). The result once again follows after some further elementary manipulations. \square

Remark 3.4. Several identities on Slater's list [17] follow as special cases of (3.3). We summarize these in the following table.

Replace q by	set z to	to obtain Slater's identity
q	$-q$	(22)
q^2	$-q$	(27) = (87)
q	iq	(28)
q^3	q^2	(40)
q^3	q	(41)
q^4	q^3	(55) = (57)
q	$e^{2\pi i/3} q$	(92)

We next give new proofs of Andrews' q -analogue of Gauss's ${}_2F_1(1/2)$ sum and a special case ($b \rightarrow \infty$) of Heine's [8] q -analogue of Gauss's sum:

$$\sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(c, q; q)_n} \left(\frac{c}{ab}\right)^n = \frac{(c/a, c/b; q)_{\infty}}{(c, c/ab; q)_{\infty}}. \quad (3.4)$$

We first recall Jackson's summation formula for a very-well-poised ${}_6\phi_5$ series [7, p. 356, Eq. (II. 20)] (which follows upon setting $e = a$ in (1.1)):

$$\sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, d; q)_n}{(q, \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}; q)_n} \left(\frac{aq}{bcd}\right)^n = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_{\infty}}{(aq/b, aq/c, aq/d, aq/bcd; q)_{\infty}}. \quad (3.5)$$

Theorem 3.5.

$$\sum_{n=0}^{\infty} \frac{(a, b; q)_n q^{n(n+1)/2}}{(q; q)_n (abq; q^2)_n} = \frac{(aq, bq; q^2)_{\infty}}{(q, abq; q^2)_{\infty}}, \quad (\text{Andrews, [1]}) \quad (3.6)$$

$$\sum_{n=0}^{\infty} \frac{(a; q)_n q^{n(n-1)/2} (-c)^n}{(c, q; q)_n a^n} = \frac{(c/a; q)_{\infty}}{(c; q)_{\infty}}, \quad (\text{Heine, [8]}) \quad (3.7)$$

Proof. Let $d \rightarrow 0$ in the Bailey pair at (2.47) to get the pair

$$\alpha_{2r} = \frac{1 - aq^{4r}}{1 - a} \frac{(a; q^2)_r (-1)^r q^{r^2-r}}{(q^2; q^2)_r}, \quad (3.8)$$

$$\alpha_{2r-1} = 0,$$

$$\beta_n = \frac{q^{n(n-1)/2}}{(aq; q^2)_n (q; q)_n}, \quad \text{with respect to } a = a.$$

Substitute this pair into (2.5), after setting $y = a/z$, to get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(a/z, z; q)_n q^{n(n+1)/2}}{(aq; q^2)_n (q; q)_n} &= \frac{(zq, aq/z; q)_{\infty}}{(aq, q; q)_{\infty}} \sum_{r=0}^{\infty} \frac{1 - aq^{4r}}{1 - a} \frac{(a, a/z, z; q^2)_r}{(zq^2, aq^2/z, q^2; q^2)_r} (-1)^r q^{r^2+r} \\ &= \frac{(zq, aq/z; q)_{\infty}}{(aq, q; q)_{\infty}} \frac{(aq^2, q^2; q^2)_{\infty}}{(q^2z, aq^2/z; q^2)_{\infty}} \\ &= \frac{(zq, aq/z; q^2)_{\infty}}{(aq, q; q^2)_{\infty}}. \end{aligned}$$

The next-to-last equality follows from (3.5) (replace q with q^2 , let $d \rightarrow \infty$, set $b = a/z$ and $c = z$), and (3.6) follows upon replacing a with ab and z with b .

Next, substitute the pair at (3.8) once again into (2.5), this time setting $y = \sqrt{aq}$. This gives

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(z; q)_n q^{n(n-1)/2}}{(-\sqrt{aq}; q)_n (q; q)_n} \left(\frac{\sqrt{aq}}{z} \right)^n &= \frac{(\sqrt{aq}, aq/z; q)_{\infty}}{(aq, \sqrt{aq}/z; q)_{\infty}} \sum_{r=0}^{\infty} \frac{1 - aq^{4r}}{1 - a} \frac{(a, z, zq; q^2)_r q^{r^2+r}}{(aq^2/z, aq^3/z, q^2; q^2)_r} \left(\frac{-a}{z^2q} \right)^r \\ &= \frac{(\sqrt{aq}, aq/z; q)_{\infty}}{(aq, \sqrt{aq}/z; q)_{\infty}} \frac{(aq^2, aq/z^2; q^2)_{\infty}}{(aq^2/z, aq/z; q^2)_{\infty}} \\ &= \frac{(-\sqrt{aq}/z; q)_{\infty}}{(-\sqrt{aq}; q)_{\infty}}. \end{aligned}$$

The next-to-last equality follows once again from (3.5) (replace q with q^2 , let $d \rightarrow \infty$, set $b = z$ and $c = zq$), and (3.7) follows upon replacing $-\sqrt{aq}$ with c and z with a . \square

3.2 A particular case of the Bailey Transform, I

If we set $y = -\sqrt{aq}$ and $z = \sqrt{aq}$ in (2.5), the following identity results, providing both series converge:

$$\sum_{n=0}^{\infty} (aq; q^2)_n (-1)^n \beta_n = \frac{1}{(aq^2; q^2)_{\infty} (-1; q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n \alpha_n. \quad (3.9)$$

This particular case was not used by Slater [17], and may possibly be regarded as being of lesser importance for two reasons. Firstly, it is certainly the case the both series above will not converge for all Bailey pairs (α_n, β_n) , and secondly, even if a series-product identity does result, the power of q on the series side may not be quadratic in the exponent (and many do not consider such identities as being of Rogers-Ramanujan-Slater type), or else the resulting identity is an easy consequence of a more general identity. However we believe such identities are sufficiently interesting to include some examples.

Corollary 3.6.

$$1 + \sum_{n=1}^{\infty} \frac{(-q^2; q^2)_{n-1} q^n}{(q^2; q^2)_n} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} (-q^6, -q^{10}, q^{16}; q^{16})_{\infty}, \quad (3.10)$$

$$\sum_{n=1}^{\infty} \frac{(q; q^2)_n q^n}{(-q; q^2)_{n+1}} = 1 + \sum_{r=1}^{\infty} q^{8r^2} (q^{4r} - q^{-4r}), \quad (3.11)$$

$$\sum_{n=0}^{\infty} \frac{(-q^2; q^2)_n q^n}{(q^2; q^2)_{n+1}} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} (-q^2, -q^{14}, q^{16}; q^{16})_{\infty}, \quad (3.12)$$

$$\sum_{n=0}^{\infty} \frac{(q^3; q^3)_n (-q)^n}{(q^2; q^2)_{n+1} (q; q)_n} = \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \frac{(q^{18}; q^{18})_{\infty}}{(q^9; q^{18})_{\infty}}. \quad (3.13)$$

Proof. For the first three identities above, insert the Bailey pairs at (2.62), (2.64) and (2.66), respectively into (3.9) (noting that $a = 1, q, q^2$, respectively), and in each case the result follows after some elementary manipulation of the resulting identity.

For (3.13), substitute the Bailey pair at (2.28) into (3.9), set $a = q^2$, and simplify the resulting identity. \square

3.3 A particular case of the Bailey Transform, II

If we set $y = q\sqrt{a}$ and let $z \rightarrow \infty$ in (2.5), the following identity results:

$$\sum_{n=0}^{\infty} (q\sqrt{a}; q)_n (-\sqrt{a})^n q^{n(n-1)/2} \beta_n = \frac{(q\sqrt{a}; q)_{\infty}}{(aq; q)_{\infty}} \sum_{n=0}^{\infty} (1 - \sqrt{a}q^n) (-\sqrt{a})^n q^{n(n-1)/2} \alpha_n. \quad (3.14)$$

This transformation also gives rise to a number of interesting false theta series identities, and we give several examples below. See §3.5 below for an explanation of false theta series.

Corollary 3.7.

$$\sum_{n=0}^{\infty} \frac{(1 - q^{n+1})(q^3; q^3)_n (-1)^n q^{n(n+1)/2}}{(q; q)_{2n+2}} = \sum_{r=0}^{\infty} q^{9r^2+3r} (1 - q^{12r+6}) + \sum_{r=0}^{\infty} q^{9r^2+12r+4} (1 - q^{-6r-3}) \quad (3.15)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n}}{(-q; q)_{2n}} = \sum_{r=0}^{\infty} q^{10r^2+r} (1 - q^{18r+9}) + \sum_{r=0}^{\infty} q^{10r^2+11r+3} (1 - q^{-2r-1}). \quad (3.16)$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(q; q)_{n+1} (-q^2; q^2)_n (-1)^n q^{(n^2+n)/2}}{(q; q)_{2n+2}} &= \sum_{r=0}^{\infty} q^{16r^2+4r} (1 - q^{24r+12}) \\ &\quad - \sum_{r=0}^{\infty} q^{16r^2+8r+1} (1 - q^{16r+8}) + \sum_{r=0}^{\infty} q^{16r^2+12r+2} (1 - q^{8r+4}). \end{aligned} \quad (3.17)$$

$$\sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(n+1)/2} (-1)^n}{(q; q^2)_{n+1}} = \sum_{r=0}^{\infty} (-1)^r q^{r^2+r}. \quad (3.18)$$

Proof. Let $a = q^2$ in the Bailey pair at (2.27), substitute the resulting pair into (3.14) (with $a = q^2$) and (3.15) follows after simplifying the left side and re-indexing two of the resulting series on the right side.

For (3.16), let $e \rightarrow \infty$ and set $d = q^{3/2}$ in the Bailey pair at (2.52), substitute the resulting pair (Slater's pair **I(4)**) into (3.14) (with $a = 1$), replace q with q^2 and simplify.

For (3.17), substitute pair (2.61) into (3.14) (with $a = 1$), and rearrange.

To get (3.18), let $a \rightarrow 1$ and set $c = q^{1/2}$ and $b = -q^{1/2}$ in Slater's general Bailey pair at (2.4), substitute the resulting pair (Slater's pair **H(1)**, corrected) into (3.14) (again with $a = 1$) and rearrange the resulting identity. \square

Remark 3.8. The transformation at (3.14) does lead to identities of Rogers-Ramanujan type, but those we found were either not new, or were simple linear combinations of existing identities.

3.4 Miscellaneous Identities of the Rogers-Ramanujan-Slater type

Finally, we exhibit some identities that follow from Bailey pairs not listed by Slater, pairs that do follow however from specializing the parameters in our general Bailey pairs. We use two cases of the transformation at (2.5) which were used by Slater. Firstly, let $y, z \rightarrow \infty$ to get

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n(a, q) = \frac{1}{(aq; q)_{\infty}} \sum_{n=0}^{\infty} a^n q^{n^2} \alpha_n(a, q). \quad (3.19)$$

Secondly, let $z \rightarrow \infty$, set $y = -\sqrt{aq}$ and then replace a with a^2 and q with q^2 to get

$$\sum_{n=0}^{\infty} (-aq; q^2)_n a^n q^{n^2} \beta_n(a^2, q^2) = \frac{(-aq; q^2)_{\infty}}{(a^2q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} a^n q^{n^2} \alpha_n(a^2, q^2). \quad (3.20)$$

Corollary 3.9.

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2} (-1)^n}{(1 + q^{2n-1})(-1; q)_{n-1} (q; q)_n} = \frac{1 + (q^4; q^4)_{\infty}}{(-1; q)_{\infty}}. \quad (3.21)$$

$$1 + \sum_{n=1}^{\infty} \frac{(q; -q^2)_n q^{n^2} (-1)^{\lfloor (n+1)/2 \rfloor}}{(1 - q^{4n-2})(-1; -q^2)_{n-1} (-q^2; -q^2)_n} = \frac{(q, -q^3; q^4)_{\infty}}{(-1, q^2; q^4)_{\infty}} [1 + (-q^4, -q^{12}, q^{16}; q^{16})]. \quad (3.22)$$

$$\sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2}(-1)^n}{(1-q^{2n+1})(q; q)_n} = (-q^3, -q^5, q^8; q^8)_{\infty}. \quad (3.23)$$

$$\sum_{n=0}^{\infty} \frac{q^{(n^2+3n)/2}(-1)^n}{(1-q^{2n+1})(q; q)_n} = (-q, -q^7, q^8; q^8)_{\infty}. \quad (3.24)$$

$$\begin{aligned} \frac{1+q^3}{(1-q)(1-q^2)} + \frac{1-q^4}{1-q} \sum_{n=1}^{\infty} \frac{q^{n^2+3n}(-q^2; q^2)_{n-1}}{(q; q)_{2n+2}} &= \frac{1}{(q; q)_{\infty}} \\ &\times \left[(-q^{18}, -q^{30}, q^{48}; q^{48})_{\infty} + q^3(-q^{10}, -q^{38}, q^{48}; q^{48})_{\infty} \right. \\ &\quad \left. - q^6(-q^2, -q^{46}, q^{48}; q^{48})_{\infty} - q^3(-q^6, -q^{42}, q^{48}; q^{48})_{\infty} \right] \end{aligned} \quad (3.25)$$

Proof. Let $d = q^2$ and $a = -1$ in the Bailey pair at (2.47). The identity at (3.21) follows upon substituting the resulting pair into (3.19) and simplifying.

For (3.22), set $d = q^2$ and $a = i$ in the pair at (2.47), substitute the resulting pair into (3.20), replace q with iq , and rearrange.

If we set $a = -q$ and $d = q^2$ in the pair at (2.48), we get (3.23) when this pair is inserted in (3.20), after replacing q with $q^{1/2}$.

The identity at (3.24) follows upon inserting the pair formed by setting $a = -q$ and $d = q^2$ in the Bailey pair at (2.49) into (3.20), and then once again replacing q with $q^{1/2}$.

Let $v \rightarrow \infty$ and set $u = iq^{3/2}$ in the Bailey pair at (2.59). This gives the Bailey pair (with respect to $a = q^2$), $\alpha_0 = \beta_0 = 1$,

$$\begin{aligned} \alpha_{4r} &= \frac{1+q^{8r+3}}{1+q^3} q^{8r^2-2r} \\ \alpha_{4r+1} &= -\frac{1+q^{8r+3}}{1+q^3} q^{8r^2+6r+3} - \frac{1+q^{8r+5}}{1+q^3} q^{8r^2+2r} \\ \alpha_{4r-1} &= 0 \\ \alpha_{4r-2} &= \frac{1+q^{8r-3}}{1+q^3} q^{8r^2-6r+3} \\ \beta_n &= \frac{q^n(1+q)(1+q^2)(-q^2; q^2)_{n-1}}{(1+q^3)(q^3; q)_{2n}}. \end{aligned}$$

Note that the expression for β_n follows from the fact that, with the stated values for u and v ,

$$K_8(n) = \frac{(1+q)(1+q^2)q^n}{(1+q^3)(1+q^{2n})}.$$

The identity at (3.25) follows upon inserting this Bailey pair in (3.19), and using the Jacobi Triple Product identity to sum pairs of series. \square

3.5 Theta-False Theta “hybrid” identities.

Rogers [13] referred to a series of the form

$$\sum_{n=-\infty}^{\infty} (\pm 1)^n q^{rn^2+sn} = \sum_{n=0}^{\infty} (\pm 1)^n q^{rn^2+sn} + \sum_{n=1}^{\infty} (\pm 1)^n q^{rn^2-sn} \quad (3.26)$$

as a *theta series of order r* , where $r \in \mathbb{Q}_+$ and $s \in \mathbb{Q}$. Due to the Jacobi triple product identity, the series (3.26) may be expressed as the infinite product

$$(\mp q^s, \mp q^{2r-s}, q^{2r}; q^{2r})_{\infty}.$$

Rogers also took interest in series of the form

$$\sum_{n=0}^{\infty} (\pm 1)^n q^{rn^2+sn} - \sum_{n=1}^{\infty} (\pm 1)^n q^{rn^2-sn} = \sum_{n=0}^{\infty} (\pm 1)^n q^{rn^2+sn} (1 - q^{2(r-s)n+(r-s)}) \quad (3.27)$$

which he called a *theta series of order r*. False theta series are not representable as infinite products, but nonetheless may have representations as Rogers-Ramanujan type q -series, and Rogers gave a number of examples of identities of false theta series. See [10, §5, p. 35 ff] for further discussion and numerous examples of false theta series identities.

Here we give several examples of identities where one side is a basic hypergeometric series and the other side comprises a theta product multiplied by a false theta series. We believe this type of identity is new.

Corollary 3.10.

$$\sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(1 - q^{2n+1})(q^2, q^2; q^2)_n} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left(1 + \sum_{r=0}^{\infty} q^{8r^2+4r+1} (1 - q^{8r+4}) \right). \quad (3.28)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1 - q^{2n+1})(q, q; q)_n} = \frac{\sum_{r=0}^{\infty} q^{6r^2+2r} (1 - q^{8r+4})}{(q; q)_{\infty}}. \quad (3.29)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(1 - q^{2n+1})(q, q; q)_n} = \frac{\sum_{r=0}^{\infty} q^{6r^2+4r} (1 - q^{4r+2})}{(q; q)_{\infty}}. \quad (3.30)$$

$$\sum_{n=0}^{\infty} \frac{(-q; q)_n q^{(n^2+n)/2}}{(1 - q^{2n+1})(q, q; q)_n} = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{r=0}^{\infty} q^{4r^2+r} (1 - q^{6r+3}). \quad (3.31)$$

$$\sum_{n=0}^{\infty} \frac{(-q; q)_n q^{(n^2+3n)/2}}{(1 - q^{2n+1})(q, q; q)_n} = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{r=0}^{\infty} q^{4r^2+3r} (1 - q^{2r+1}). \quad (3.32)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(1 - q^{2n+1})(q^2; q^2)_n} = (-q; q^2)_{\infty} \sum_{r=0}^{\infty} (-1)^r q^{6r^2+4r} (1 - q^{4r+2}). \quad (3.33)$$

$$\sum_{n=0}^{\infty} \frac{(-q; q^2)_{n+1} q^{n^2-2n}}{(q^2; q^2)_n (q^2; q^2)_{n+1}} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left[1 + q + q^2 + q^3 + \sum_{r=0}^{\infty} q^{r^2} (q^{-2r} - q^{2r}) \right]. \quad (3.34)$$

Proof. The identity at (3.28) follows upon setting $d = q^2$ and $a = q$ in the pair at (2.48), substituting the resulting pair (Slater's pair **I(10)**) into (3.14) (with $a = q$, replacing q with q^2 , and finally q with $-q$).

The identity at (3.29) follows from setting $d = q^2$ and $a = q$ in the Bailey pair at (2.48), and substituting the resulting pair into (3.19), while the identity at (3.31) follows from substituting the same pair in (3.20), and replacing q with $q^{1/2}$.

Likewise, the identity at (3.30) follows as a consequence setting $d = q^2$ and $a = q$ in the Bailey pair at (2.49), and substituting the resulting pair into (3.19), while for (3.33) we proceed similarly, after creating a Bailey pair by instead setting $a = -q$ (and keeping $d = q^2$).

The identity at (3.32) follows upon inserting the pair used in the proof of (3.30) into (3.20), after replacing q with $q^{1/2}$.

For (3.34), insert Slater's pair **H(12)** ($a = q$, $c = q$ and $b \rightarrow 0$ in (2.4)) into (3.14) (with $a = q$), replace q with q^2 , and rearrange. □

4 Concluding Remarks

In a paper by the second author [15], it was shown that more than half of Slater's identities could be derived from just three general Bailey pairs together with several limiting cases of Bailey's lemma and an associated family of q -difference equations. Here, we attempt to put Slater's work in a broader context via general Bailey pairs, but without the use of q -difference equations. Both approaches have their merits, and both yield new identities. It may well be worth exploring the combinatorial consequences of the new identities presented here.

We also note that we have given just a sample of the identities that may be produced by the methods used in Corollaries 3.6 - 3.10, and that it is likely that many similar identities may be produced by employing other Bailey pairs.

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$$1 + \frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)} \cdot x + \frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^\gamma)(1-q^{\gamma+1})} \cdot x^2 + \dots,$$

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