SOME APPLICATIONS OF A BAILEY-TYPE TRANSFORMATION

JAMES MC LAUGHLIN AND PETER ZIMMER

Abstract. If k is set equal to aq in the definition of a WP Bailey pair,

$$\beta_n(a,k)=\sum_{j=0}^{n}\frac{(k/a)_{n-j}(k)_{n+j}}{(q)_{n-j}(aq)_{n+j}}\alpha_j(a,k),$$

this equation reduces to $\beta_n=\sum_{j=0}^{n}\alpha_j$.

This seemingly trivial relation connecting the α_n's with the β_n's has some interesting consequences, including several basic hypergeometric summation formulae, a connection to the Prouhet-Tarry-Escott problem, some new identities of the Rogers-Ramanujan-Slater type, some new expressions for false theta series as basic hypergeometric series, and new transformation formulae for poly-basic hypergeometric series.

1. Introduction

We begin by recalling a construction of Andrews [1]. If a pair of sequences $(\alpha_n(a,k),\ \beta_n(a,k))$ satisfy

$$\beta_n(a,k)=\sum_{j=0}^{n}\frac{(k/a)_{n-j}(k)_{n+j}}{(q)_{n-j}(aq)_{n+j}}\alpha_j(a,k), \tag{1.1}$$

then so does the pair $(\alpha'_n(a,k),\ \beta'_n(a,k))$, where

$$\begin{aligned}\alpha'_n(a,k)&=\frac{(\rho_1,\rho_2)_n}{(aq/\rho_1,aq/\rho_2)_n}\left(\frac{k}{c}\right)^n\alpha_n(a,c),\\ \beta'_n(a,k)&=\frac{(k\rho_1/a,k\rho_2/a)_n}{(aq/\rho_1,aq/\rho_2)_n}\\ &\quad\times\sum_{j=0}^{n}\frac{(1-cq^{2j})(\rho_1,\rho_2)_j(k/c)_{n-j}(k)_{n+j}}{(1-c)(k\rho_1/a,k\rho_2/a)_n(q)_{n-j}(qc)_{n+j}}\left(\frac{k}{c}\right)^j\beta_j(a,c),\end{aligned} \tag{1.2}$$

with $c=k\rho_1\rho_2/aq$. A pair of sequences satisfying (1.1) is termed a *WP-Bailey pair*. If $k=0$, the pair of sequences become what is termed a *Bailey pair relative to a*.

Date: November 14, 2008.

2000 *Mathematics Subject Classification.* Primary: 33D15. Secondary:11B65, 05A19.

Key words and phrases. Q-Series, Rogers-Ramanujan Type Identities, Bailey chains, Prouhet-Tarry-Escott Problem.

Bailey [4, 5] used the q-Gauss sum,

$$(1.3)\qquad {}_2\phi_1(a,b;c;q,c/ab)=\frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty},$$

to get that, if (α_n,β_n) are a Bailey pair relative to a, then
(1.4)

$$\sum_{n=0}^{\infty}(y,z;q)_n\left(\frac{aq}{yz}\right)^n\beta_n=\frac{(aq/y,aq/z;q)_\infty}{(aq,aq/yz;q)_\infty}\sum_{n=0}^{\infty}\frac{(y,z;q)_n}{(aq/y,aq/z;q)_n}\left(\frac{x}{yz}\right)^n\alpha_n.$$

Slater, in [21] and [22], subsequently used this transformation of Bailey to derive 130 identities of the Rogers-Ramanujan type.

The first major variations in Bailey's construct at (1.4) appear to be due to Bressoud [7]. Another variation was given by Singh in [20]. All of these variations were put in a more formal setting by Andrews in [1], where he introduced the generalization of the standard Bailey pair defined above.

In the same paper Andrews also described a second way to construct a new WP-Bailey pair from a given pair. These two constructions allowed a "tree" of WP-Bailey pairs to be generated from a single WP-Bailey pair. These two branches of the WP-Bailey tree were further investigated by Andrews and Berkovich in [2]. Spiridonov [23] derived an elliptic generalization of Andrews first WP-Bailey chain, and Warnaar [25] added four new branches to the WP-Bailey tree, two of which had generalizations to the elliptic level. More recently, and motivated in part by the papers above, Liu and Ma [14] introduced the idea of a general WP-Bailey chain (as a solution to a system of linear equations), and added one new branch to the WP-Bailey tree.

As we might expect, Andrews generalization of a Bailey pair leads to a generalization of (1.4). Indeed Andrews WP-Bailey chain at (1.2) can easily be shown to imply the following result (substitute the expression for $\alpha_n'(a,k)$ in (1.1), set the two expressions for $\beta_n'(a,k)$ equal, and let $n\to\infty$). Note that setting $k=0$ recovers Bailey's transformation at (1.4). (We initially derived (1.6) in a way similar to Bailey's derivation of (1.4), before realizing that it followed from Andrews' construction (1.2).)

Theorem 1. *Under suitable convergence conditions, if* $(\alpha_n(a,k),\beta_n(a,k))$ *satisfy*

$$(1.5)\qquad \beta_n(a,k)=\sum_{j=0}^{n}\frac{(k/a)_{n-j}(k)_{n+j}}{(q)_{n-j}(aq)_{n+j}}\alpha_j(a,k),$$

then

$$(1.6)\qquad \sum_{n=0}^{\infty}\frac{(1-kq^{2n})(\rho_1,\rho_2;q)_n}{(1-k)(kq/\rho_1,kq/\rho_2;q)_n}\left(\frac{aq}{\rho_1\rho_2}\right)^n\beta_n(a,k)=$$

$$\frac{(kq,kq/\rho_1\rho_2,aq/\rho_1,aq/\rho_2;q)_\infty}{(kq/\rho_1,kq/\rho_2,aq/\rho_1\rho_2,aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2;q)_n}\left(\frac{aq}{\rho_1\rho_2}\right)^n\alpha_n(a,k).$$

In the present paper we investigate what at first glance may appear to be a trivial special case of Theorem 1.

Corollary 1. *If* $\beta_n = \sum_{r=0}^{n} \alpha_r$*, then assuming both series converge,*

$$\sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n x^n \beta_n}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} = \frac{(1-xy)\,(1-xz)}{(1-x)\,(1-xyz)} \sum_{n=0}^{\infty} \frac{(y,z;q)_n x^n \alpha_n}{(xy,xz;q)_n}. \tag{1.7}$$

Proof. Let $k = xyz$, $a = xyz/q$, $\rho_1 = y$ and $\rho_2 = z$ in Theorem 1. □

This seemingly trivial relation connecting the α_n's with the β_n's has some interesting consequences, including several basic hypergeometric summation formulae, a connection to the Prouhet-Tarry-Escott problem, some new identities of the Rogers-Ramanujan-Slater type, some new expressions for false theta series as basic hypergeometric series, and new transformation formulae for poly-basic hypergeometric series.

We employ the usual notations. Let a and q be complex numbers, with $|q| < 1$ unless otherwise stated. Then

$$(a)_0 = (a;q)_0 := 1, \qquad (a)_n = (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j), \text{ for } n \in \mathbb{N},$$
$$(a_1;q)_n(a_2;q)_n \ldots (a_k;q)_n = (a_1, a_2, \ldots, a_k; q)_n,$$
$$(a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j),$$
$$(a_1;q)_\infty(a_2;q)_\infty \ldots (a_k;q)_\infty = (a_1, a_2, \ldots, a_k; q)_\infty.$$

An ${}_r\phi_s$ basic hypergeometric series is defined by

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, x \right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n \ldots (a_r;q)_n}{(q;q)_n(b_1;q)_n \ldots (b_s;q)_n} \left((-1)^n q^{n(n-1)/2}\right)^{s+1-r} x^n.$$

For future use we also recall the q-binomial theorem,

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{1.8}$$

2. Various Summation Formulae for Basic Hypergeometric series

We next derive a number of transformation formulae for basic hypergeometric series, transformations that give rise to summation formulae for particular choices of the parameters.

Corollary 2. *For q and x inside the unit disc,*

$$\sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_{2n} x^{2n}}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_{2n}} = \frac{(1-xy)\,(1-xz)}{(1-x)\,(1-xyz)} \sum_{n=0}^{\infty} \frac{(y, z; q)_n (-x)^n}{(xy, xz; q)_n}. \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{(1-q^{2n+1}/x)(q/x^2; q)_{2n} x^{2n}}{(q;q)_{2n+1}} = \frac{1}{1+x} \frac{(q/x;q)_\infty}{(x;q)_\infty}, \quad x \neq 0. \tag{2.2}$$

Proof. In Corollary 1 let $\alpha_r = (-1)^r$ to get (2.1). Then set $y = q/x$, $z = q/x^2$, apply (1.8) to the right side, replace x by $-x$ and (2.2) follows. □

Corollary 3. *For q and x inside the unit disc,*

$$\sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n x^n (n+1)}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} = \frac{(1-xy)\,(1-xz)}{(1-x)\,(1-xyz)} \sum_{n=0}^{\infty} \frac{(y, z; q)_n x^n}{(xy, xz; q)_n}. \tag{2.3}$$

$$\sum_{n=0}^{\infty} \frac{(1+q^{n+1}/x)(q/x^2; q)_n x^n (n+1)}{(q;q)_{n+1}} = \frac{1}{1-x} \frac{(q/x;q)_\infty}{(x;q)_\infty}. \tag{2.4}$$

Proof. Set $\alpha_n = 1$ in Corollary 1 to get (2.3). The identity at (2.4) follows from (2.3) upon setting $y = q/x^2$, $z = q/x$, using (1.8) to sum the right side and then simplifying. □

Corollary 4. *For q, x and u all inside the unit disc,*

$$\sum_{n=0}^{\infty} \frac{(1+q^{n+1}/x)(q/x^2; q)_n x^n (1-u^{n+1})}{(q;q)_{n+1}} = \frac{1-u}{1-x} \frac{(qu/x;q)_\infty}{(xu;q)_\infty}. \tag{2.5}$$

Proof. Set $\alpha_n = u^n$, $y = q/x$ and $z = q/x^2$. Now apply the q-binomial theorem (1.8) to the right side. □

Corollary 5.

(2.6)

$${}_5\phi_4 \left[\begin{matrix} q\sqrt{xyz}, -q\sqrt{xyz}, y, z, cq \\ \sqrt{xyz}, -\sqrt{xyz}, qxy, qxz \end{matrix} ; q, x \right] = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \, {}_3\phi_2 \left[\begin{matrix} y, z, c \\ xy, xz \end{matrix} ; q, xq \right].$$

$${}_3\phi_2 \left[\begin{matrix} -qxy, y, x \\ -xy, qx^2y \end{matrix} ; q, x \right] = \frac{1}{1+xy} \frac{(x^2, qxy; q)_\infty}{(qx^2y, x; q)_\infty}. \tag{2.7}$$

Proof. We define $\alpha_0 = 1$, and for $n > 0$,

$$\alpha_n = \frac{(cq;q)_n}{(q;q)_n} - \frac{(cq;q)_{n-1}}{(q;q)_{n-1}} = \frac{(c;q)_n}{(q;q)_n} q^n.$$

Substitution into (1.7) immediately gives (2.6). Equation (2.7) follows upon letting $c = x/q$, $z = xy$ and using (1.3) to sum the resulting right side and simplifying. □

3. Transformation Formulae for basic- and polybasic Hypergeometric Series

In contrast to the situation with basic hypergeometric series, most (possibly all) summation formulae for poly-basic hypergeometric series arise because the series involved telescope. This means that the terms in such an identity may be inserted in (1.7) to produce a transformation formula for polybasic hypergeometric series containing an additional base. Setting all the bases equal to q^m, for some integer m, then gives a transformation formula for basic hypergeometric series. We give one example in the next corollary, which contains a transformation formula connecting polybasic hypergeometric series with five independent bases.

Corollary 6. *Let P, p, Q, q, R and x all lie inside the unit disc, and let a, b, c, y and z be complex numbers such that the denominators below are bounded away from zero. Then*

$$(3.1)\quad \sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} \frac{(ap^2;p^2)_n \left(bP^2;P^2\right)_n}{\left(\frac{PQR}{p};\frac{PQR}{p}\right)_n \left(\frac{apPQ}{cR};\frac{pPQ}{R}\right)_n}$$

$$\times \frac{(cR^2;R^2)_n \left(\frac{aQ^2}{bc};Q^2\right)_n}{\left(\frac{apQR}{bP};\frac{pQR}{P}\right)_n \left(\frac{bcpPR}{Q};\frac{pPR}{Q}\right)_n} x^n$$

$$= \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \times$$

$$\sum_{n=0}^{\infty} \frac{(y,z;q)_n}{(xy,xz;q)_n} \frac{(1-ap^nP^nQ^nR^n)\left(1-b\frac{p^nP^n}{Q^nR^n}\right)\left(1-\frac{P^nQ^n}{cp^nR^n}\right)\left(1-\frac{ap^nQ^n}{bcP^nR^n}\right)}{(1-a)(1-b)\left(1-\frac{1}{c}\right)\left(1-\frac{a}{bc}\right)}$$

$$\times \frac{(a;p^2)_n \left(b;P^2\right)_n}{\left(\frac{PQR}{p};\frac{PQR}{p}\right)_n \left(\frac{apPQ}{cR};\frac{pPQ}{R}\right)_n} \frac{(c;R^2)_n \left(\frac{a}{bc};Q^2\right)_n}{\left(\frac{apQR}{bP};\frac{pQR}{P}\right)_n \left(\frac{bcpPR}{Q};\frac{pPR}{Q}\right)_n} (xR^2)^n;$$

$$(3.2)\quad \sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} \frac{(aq^m, bq^m, cq^m, \frac{aq^m}{bc}; q^m)_n}{\left(\frac{a}{c}q^m, \frac{a}{b}q^m, bcq^m, q^m; q^m\right)_n} x^n$$

$$= \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \times$$

$$\sum_{n=0}^{\infty} \frac{(y,z;q)_n}{(xy,xz;q)_n} \frac{(q^m\sqrt{a},-q^m\sqrt{a},a,b,c,\frac{a}{bc};q^m)_n}{\left(\sqrt{a},-\sqrt{a},\frac{a}{c}q^m,\frac{a}{b}q^m,bcq^m,q^m;q^m\right)_n}(xq^m)^n.$$

Proof. We use the special case $m=0$, $d=1$ of the identity of Subbarao and Verma labeled (2.2) in [24], namely,

$$\sum_{k=0}^{n} \frac{(1-ap^kP^kQ^kR^k)\left(1-b\frac{p^kP^k}{Q^kR^k}\right)\left(1-\frac{P^kQ^k}{cp^kR^k}\right)\left(1-\frac{ap^kQ^k}{bcP^kR^k}\right)}{(1-a)(1-b)\left(1-\frac{1}{c}\right)\left(1-\frac{a}{bc}\right)} \tag{3.3}$$
$$\times \frac{(a;p^2)_k\left(b;P^2\right)_k}{\left(\frac{PQR}{p};\frac{PQR}{p}\right)_k\left(\frac{apPQ}{cR};\frac{pPQ}{R}\right)_k} \frac{(c;R^2)_k\left(\frac{a}{bc};Q^2\right)_k}{\left(\frac{apQR}{bP};\frac{pQR}{P}\right)_k\left(\frac{bcpPR}{Q};\frac{pPR}{Q}\right)_k} R^{2k}$$
$$= \frac{(ap^2;p^2)_n\left(bP^2;P^2\right)_n(cR^2;R^2)_n\left(\frac{aQ^2}{bc};Q^2\right)_n}{\left(\frac{PQR}{p};\frac{PQR}{p}\right)_n\left(\frac{apPQ}{cR};\frac{pPQ}{R}\right)_n\left(\frac{apQR}{bP};\frac{pQR}{P}\right)_n\left(\frac{bcpPR}{Q};\frac{pPR}{Q}\right)_n},$$

and then in (1.7) let α_i be the i-th term in the sum above, and let β_n be the quantity on the right side above.

The identity at (3.2) follows upon setting $P=Q=p=R=q^{m/2}$ and simplifying. □

4. A Connection with the Prouhet-Tarry-Escott Problem

We begin with a simple example.

Corollary 7.

$$ {}_6\phi_5\left[\begin{matrix} q\sqrt{xyz},-q\sqrt{xyz},y,z,aq,bq \\ \sqrt{xyz},-\sqrt{xyz},qxy,qxz,abq \end{matrix};q,x\right] = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)}\,{}_4\phi_3\left[\begin{matrix} y,z,a,b \\ xy,xz,abq \end{matrix};q,xq\right]. \tag{4.1}$$

Proof. This time, in Corollary 1, define $\alpha_0=1$, and for $n>0$,

$$\alpha_n = \frac{(aq,bq;q)_n}{(abq,q;q)_n} - \frac{(aq,bq;q)_{n-1}}{(abq,q;q)_{n-1}} = \frac{(a,b;q)_nq^n}{(abq,q;q)_n}.$$

The result follows as above. □

The telescoping approach used in Corollary 7 can be generalized in one direction. We have the following result.

Proposition 1. *Let x, y and q be complex numbers with $|x|$, $|q|<1$. Suppose $a_1,a_2,\dots,a_m$ are non-zero complex numbers and let $b_1,b_2,\dots,b_{m-1}$ satisfy*

$$(z-1)\prod_{i=1}^{m-1}(z-b_i) = \prod_{i=1}^{m}(z-a_i) - \prod_{i=1}^{m}(1-a_i). \tag{4.2}$$

Suppose further that $b_i \neq 0$, for $1 \leq i \leq m-1$. Then

$$(4.3)\quad {}_{m+4}\phi_{m+3}\left[\begin{matrix} q\sqrt{xyz},\ -q\sqrt{xyz},\ y,\ z,\ a_1q,\ \ldots,\ a_{m-1}q,\ a_mq \\ \sqrt{xyz},\ -\sqrt{xyz},\ qxy,\ qxz,\ b_1q,\ \ldots,\ b_{m-1}q \end{matrix};q,\ x\right]$$
$$= \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)}\,{}_{m+2}\phi_{m+1}\left[\begin{matrix} y,\ z,\ a_1,\ \ldots,\ a_{m-1},\ a_m \\ xy,\ xz,\ b_1q,\ \ldots,\ b_{m-1}q \end{matrix};q,\ xq^m\right].$$

Proof. Define $\alpha_0 = 1$, and for $n \geq 1$, set

$$\alpha_n = \frac{(a_1q, a_2q, \ldots, a_{m-1}q, a_mq; q)_n}{(b_1q, b_2q, \ldots, b_{m-1}q,\ q; q)_n} - \frac{(a_1q, a_2q, \ldots, a_{m-1}q, a_mq; q)_{n-1}}{(b_1q, b_2q, \ldots, b_{m-1}q,\ q; q)_{n-1}}.$$

By (4.2),

$$\alpha_n = \frac{(a_1, a_2, \ldots, a_{m-1}, a_m; q)_n}{(b_1q, b_2q, \ldots, b_{m-1}q,\ q; q)_n}\, q^{mn}$$

and clearly

$$(4.4)\qquad \beta_n = \sum_{r=0}^{n} \alpha_r = \frac{(a_1q, a_2q, \ldots, a_{m-1}q, a_mq; q)_n}{(b_1q, b_2q, \ldots, b_{m-1}q,\ q; q)_n}.$$

The result follows from Corollary 1. □

The fundamental theorem of algebra guarantees that there is no shortage of sets of complex numbers $a_1, a_2, \ldots, a_m,\ b_1, b_2, \ldots, b_{m-1}$ satisfying (4.2), but for $m > 4$ it is still a problem to find explicit examples. However, a related problem in number theory provides solutions for $m \leq 10$ and $m = 12$.

The *Prouhet-Tarry-Escott* problem asks for two distinct multisets of integers $A = \{a_1, ..., a_m\}$ and $B = \{b_1, ..., b_m\}$ such that

$$(4.5)\qquad \sum_{i=1}^{m} a_i^e = \sum_{i=1}^{m} b_i^e, \text{ for } e = 1, 2, \ldots, k,$$

for some integer $k < m$. If $k = m-1$, such a solution is called *ideal.* We write

$$(4.6)\qquad \{a_1, ..., a_m\} \stackrel{k}{=} \{b_1, ..., b_m\}$$

to denote a solution to the Prouhet-Tarry-Escott problem.

The connection between the Prouhet-Tarry-Escott problem and the problem mentioned above is contained in the following proposition (see [6], page 2065).

Proposition 2. *The multisets $A = \{a_1, ..., a_m\}$ and $B = \{b_1, ..., b_m\}$ form an ideal solution to the Prouhet-Tarry-Escott problem if and only if*

$$\prod_{i=1}^{m}(z-a_i) - \prod_{i=1}^{m}(z-b_i) = C,$$

for some constant C.

Note that the fact that $b_m = 1$ is not a problem, since if

$$\{a_1, ..., a_m\} \stackrel{m-1}{=} \{b_1, ..., b_m\},$$

then

$$\{Ma_1 + K, ..., Ma_m + K\} \stackrel{m-1}{=} \{Mb_1 + K, ..., Mb_m + K\},$$

for constants M and K (see Lemma 1 in [8], for example).

Parametric ideal solutions are known for $m = 1, \dots, 8$ and particular numerical solutions are known for $m = 9, 10$ and 12. Although every ideal solution to the Prouhet-Tarry-Escott problem gives rise to a transformation between basic hypergeometric series, we will consider just one example. Note also that it is not necessary, for our purposes, that the a_i's and b_i's be integers. As above, we assume x, y and q are complex numbers, with $|x|, |q| < 1$.

Corollary 8. *Let m and n be non-zero complex numbers. Set*

$$\begin{aligned}
a_1 &= -3m^2 + 7nm - 2n^2 + 1, & b_1 &= -3m^2 + 8nm + n^2 + 1, \\
a_2 &= -2m^2 + 8nm + 2n^2 + 1, & b_2 &= -2m^2 + 3nm - 3n^2 + 1, \\
a_3 &= -m^2 - n^2 + 1, & b_3 &= -m^2 + 10nm - n^2 + 1, \\
a_4 &= 2m^2 + 3nm + n^2 + 1, & b_4 &= 2m^2 + 2nm - 2n^2 + 1, \\
a_5 &= m^2 + 2nm - 3n^2 + 1, & b_5 &= m^2 + 7nm + 2n^2 + 1, \\
a_6 &= 10mn + 1.
\end{aligned} \tag{4.7}$$

Then

$$\begin{aligned}
&{}_{10}\phi_9 \left[\begin{matrix} q\sqrt{xyz}, \ -q\sqrt{xyz}, \ y, \ z, \ a_1q, \ a_2q, \ a_3q, \ a_4q, \ a_5q, \ a_6q \\ \sqrt{xyz}, \ -\sqrt{xyz}, \ qxy, \ qxz, \ b_1q, \ b_2q, \ b_3q, \ b_4q, \ b_5q \end{matrix} ; q, \ x \right] \\
&\quad = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \, {}_8\phi_7 \left[\begin{matrix} y, \ z, \ a_1, \ a_2, \ a_3, \ a_4, \ a_5, \ a_6 \\ xy, \ xz, \ b_1q, \ b_2q, \ b_3q, \ b_4q, \ b_5q \end{matrix} ; q, \ xq^6 \right].
\end{aligned} \tag{4.8}$$

Proof. We have from page 629–30 and Lemma 1 in [8], that if

$$\begin{aligned}
a_1 &= -5m^2 + 4nm - 3n^2 + K, & b_1 &= -5m^2 + 6nm + 3n^2 + K, \\
a_2 &= -3m^2 + 6nm + 5n^2 + K, & b_2 &= -3m^2 - 4nm - 5n^2 + K, \\
a_3 &= -m^2 - 10nm - n^2 + K, & b_3 &= -m^2 + 10nm - n^2 + K, \\
a_4 &= 5m^2 - 4nm + 3n^2 + K, & b_4 &= 5m^2 - 6nm - 3n^2 + K, \\
a_5 &= 3m^2 - 6nm - 5n^2 + K, & b_5 &= 3m^2 + 4nm + 5n^2 + K, \\
a_6 &= m^2 + 10nm + n^2 + K, & b_6 &= m^2 - 10nm + n^2 + K,
\end{aligned} \tag{4.9}$$

then

$$\{a_1, a_2, a_3, a_4, a_5, a_6\} \stackrel{5}{=} \{b_1, b_2, b_3, b_4, b_5, b_6\}.$$

We set $b_6 = 1$, solve for K and back-substitute in (4.9). We then replace m by $m/\sqrt{2}$ and n by $n/\sqrt{2}$. This leads to the values for the a_i's and b_i's given at (4.7) and the result follows, as before, from Proposition 1. □

We also note each ideal solution to the Prouhet-Tarry-Escott problem leads to an infinite summation formula, upon letting $n \to \infty$ in (4.4). We give one example.

Corollary 9. *Let m be a non-zero complex number. Set*

$$\{a_i\}_{i=1}^{12} = \{1+170m, 1+126m, 1+209m, 1+87m, 1+234m, 1+62m, \\ 1+275m, 1+21m, 1+288m, 1+8m, 1+299m, 1-3m\},$$

$$\{b_i\}_{i=1}^{11} = \{1+183m, 1+113m, 1+195m, 1+101m, 1+242m, 1+54m, \\ 1+269m, 1+27m, 1+294m, 1+2m, 1+296m\}.$$

Then

$$(4.10) \quad {}_{12}\phi_{11}\left[\begin{matrix} a_1,\, a_2,\, a_3,\, a_4,\, a_5,\, a_6,\, a_7,\, a_8, a_9, a_{10}, a_{11}, a_{12} \\ b_1q,\, b_2q,\, b_3q,\, b_4q,\, b_5q,\, b_6q, b_7q, b_9q, b_{10}q, b_{11}q \end{matrix}; q,\, q^{12}\right]. \\ = \frac{(a_1q, a_2q, a_3q, a_4q, a_5q, a_6q, a_7q, a_8q, a_9q, a_{10}q, a_{11}q, a_{12}q; q)_\infty}{(b_1q,\, b_2q,\, b_3q,\, b_4q,\, b_5q,\, b_6q, b_7q, b_9q, b_{10}q, b_{11}q, q; q)_\infty}$$

Proof. We use a result of Nuutti Kuosa, Jean-Charles Meyrignac and Chen Shuwen (see [19]), namely, that if

(4.11)

$$A = \{K+22m, K-22m, K+61m, K-61m, K+86m, K-86m, \\ K+127m, K-127m, K+140m, K-140m, K+151m, K-151m\},$$
$$B = \{K+35m, K-35m, K+47m, K-47m, K+94m, K-94m, \\ K+121m, K-121m, K+146m, K-146m, K+148m, K-148m\},$$

then

$$A \overset{11}{=} B.$$

□

Remark: Note that while the K and m are irrelevant in (4.11) in so far as finding integer solutions to the Prouhet-Tarry-Escott problem (since the solution derived another solution by scaling by m and translating by K is trivially equivalent to the original solution), solving $B_{12} = 1$ for K leaves m as a non-trivial free parameter in (4.10).

5. Identities of the Rogers-Ramanujan-Slater Type

We next prove a number of identities of the Rogers-Ramanujan-Slater type. We believe these to be new. We first prove two general transformations.

Corollary 10. *For q and x inside the unit disc, and integers $a>0$ and b,*

$$\sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n x^n q^{(an^2+bn)/2}}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n (-q^{(a+b)/2}; q^a)_n} \tag{5.1}$$
$$= \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \left(1 - q^{(a-b)/2} \sum_{n=1}^{\infty} \frac{(y,z;q)_n x^n q^{(an^2+(b-2a)n)/2}}{(xy, xz; q)_n (-q^{(a+b)/2}; q^a)_n}\right).$$

Proof. In Corollary 1 set $\alpha_0 = 1$ and, for $n > 0$,

$$\alpha_n = \frac{q^{(an^2+bn)/2}}{(-q^{(a+b)/2}; q^a)_n} - \frac{q^{(a(n-1)^2+b(n-1))/2}}{(-q^{(a+b)/2}; q^a)_{n-1}} = -q^{(a-b)/2} \frac{q^{(an^2+(b-2a)n)/2}}{(-q^{(a+b)/2}; q^a)_n}.$$

□

Corollary 11. *For q and x inside the unit disc, and integers $a>0$ and b,*

$$\sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n x^n q^{an^2+bn}}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \tag{5.2}$$
$$\times \left(1 - q^{(a-b)} \sum_{n=1}^{\infty} \frac{(y,z;q)_n x^n q^{an^2+(b-2a)n}(1-q^{2an+b-a})}{(xy, xz; q)_n}\right).$$

Proof. In Corollary 1 set $\alpha_0 = 1$ and, for $n > 0$,

$$\alpha_n = q^{an^2+bn} - q^{a(n-1)^2+b(n-1)} = -q^{an^2+(b-2a)n+a-b}(1-q^{2an+b-a}).$$

□

Corollary 12.

$$\sum_{n=0}^{\infty} \frac{(1+q^{-2n+3})q^{n^2+6n}}{(q^4;q^4)_n} = \frac{1}{(q^2,q^3;q^5)_\infty (-q^2;q^2)_\infty}. \tag{5.3}$$

$$\sum_{n=0}^{\infty} \frac{(1+q^{-2n+1})q^{n^2+4n}}{(q^4;q^4)_n} = \frac{1}{(q,q^4;q^5)_\infty (-q^2;q^2)_\infty}. \tag{5.4}$$

Proof. In (5.2), set $z=0$, replace x by x/y and let $y \to \infty$ to get

$$\sum_{n=0}^{\infty} \frac{(-x)^n q^{an^2+bn+n(n-1)/2}}{(xq;q)_n} = (1-x) \tag{5.5}$$
$$\times \left(1 - q^{(a-b)} \sum_{n=1}^{\infty} \frac{(-x)^n q^{an^2+(b-2a)n+n(n-1)/2}(1-q^{2an+b-a})}{(x;q)_n}\right).$$

Next, let $a=-1/4$, $b=1$, replace q by q^4 and let $x \to 1$ to get

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(q^4;q^4)_n} = -q^{-5} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+4n}(1-q^{-2n+5})}{(q^4;q^4)_{n-1}}.$$

Replace q by $-q$, re-index the right side by replacing n by $n+1$ and (5.3) follows from the following identity of Rogers ([17], page 331):

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n} = \frac{1}{(q^2,q^3;q^5)_\infty(-q^2;q^2)_\infty}.$$

The identity at (5.4) follows similarly, using instead $a=-1/4$, $b=1/2$ in (5.5) and employing another identity of Rogers ([17], page 330):

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} = \frac{1}{(q,q^4;q^5)_\infty(-q^2;q^2)_\infty}.$$

□

Corollary 13.

$$\sum_{n=0}^{\infty} \frac{(b,q^3/b;q)_n q^{n(n+1)/2}}{(q^2;q^2)_{n+1}(q;q)_n} = \frac{(q^4/b,bq;q^2)_\infty}{(q;q)_\infty}. \tag{5.6}$$

$$\sum_{n=0}^{\infty} \frac{(1-q^{2n+1})(-q^3;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \frac{1}{(q^3,q^4,q^5;q^8)_\infty}. \tag{5.7}$$

$$\sum_{n=0}^{\infty} \frac{(1-q^{2n-1})(-q^3;q^2)_n q^{n^2-2n}}{(q^2;q^2)_n} = \frac{1}{(q,q^4,q^7;q^8)_\infty}. \tag{5.8}$$

$$1+\sum_{n=1}^{\infty} \frac{(-q;q)_n q^{(n^2-n)/2}}{(q;q)_{n-1}} = \frac{(-1;q)_\infty(-q^6,-q^{10},q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}. \tag{5.9}$$

$$-1+\sum_{n=1}^{\infty} \frac{(-q;q)_n q^{(n^2-n)/2}}{(q;q)_{n-1}} = q\frac{(-1;q)_\infty(-q^2,-q^{14},q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}. \tag{5.10}$$

Proof. In (4.1), let $z=0$, replace x by x/y and let $y\to\infty$ to get

$$\sum_{n=0}^{\infty} \frac{(aq,bq;q)_n(-x)^n q^{n(n-1)/2}}{(qx,abq,q;q)_n} = (1-x)\sum_{n=0}^{\infty} \frac{(a,b;q)_n(-xq)^n q^{n(n-1)/2}}{(x,abq,q;q)_n}.$$

Then set $x=-q$, $a=b/q$ and then use Andrews' q-Bailey identity,

$$\sum_{n=0}^{\infty} \frac{(b,q/b;q)_n c^n q^{n(n-1)/2}}{(c;q)_n(q^2;q^2)_n} = \frac{(cq/b,bc;q^2)_\infty}{(c;q)_\infty}$$

with $c=q^2$, to sum the right side. Finally, replace b by b/q and (5.6) follows after a slight manipulation.

For the remaining identities, in (4.1) replace x by x/y, let $y\to\infty$ and then set $z=x$ and $b=0$ to get

$$\sum_{n=0}^{\infty} \frac{(1+xq^n)(aq;q)_n(-x)^n q^{n(n-1)/2}}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{(a;q)_n(-x)^n q^{n(n+1)/2}}{(q;q)_n}.$$

For (5.7) and (5.8), replace q by q^2, set $a=-q$ and, respectively, $x=-q$ and $x=-1/q$, and use the Göllnitz-Gordon-Slater identities ([12], [13], [22])

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n}=\frac{1}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty}, \tag{5.11}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n}=\frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty},$$

to sum the respective right sides.

For (5.9), set $a=x=-1$ and use the following identity of Gessel and Stanton ([11], page 196)

$$1+\sum_{n=1}^{\infty}\frac{(-q;q)_{n-1}q^{(n^2+n)/2}}{(q;q)_n}=\frac{(-q;q)_\infty(-q^6,-q^{10},q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}.$$

to sum the resulting right side. The identity at (5.10) follows similarly, again with $a=x=-1$, upon using another identity of Gessel and Stanton ([11], page 196)

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{(n^2+3n)/2}}{(q;q)_{n+1}}=\frac{(-q;q)_\infty(-q^2,-q^{14},q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}.$$

□

Corollary 14.

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_{n+1}q^{n^2+2n}}{(q;q)_{2n+3}}=2(-q^2,-q^{14},q^{16};q^{16})_\infty\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}-\frac{1}{1-q}. \tag{5.12}$$

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2}}{(q;q)_{2n+1}}=2\frac{(q^2,q^{14},q^{16};q^{16})_\infty(q^{12},q^{20};q^{32})_\infty}{(q;q)_\infty}-1. \tag{5.13}$$

Proof. We use (5.1) to prove these identities. First, let $z\to 0$ and replace q with q^2 to get

$$\sum_{n=0}^{\infty}\frac{(y;q^2)_n x^n q^{an^2+bn}}{(q^2xy;q^2)_n(-q^{a+b};q^{2a})_n} \tag{5.14}$$
$$=\frac{(1-xy)}{(1-x)}\left(1-q^{a-b}\sum_{n=1}^{\infty}\frac{(y;q^2)_n x^n q^{an^2+(b-2a)n}}{(xy;q^2)_n(-q^{a+b};q^{2a})_n}\right).$$

For (5.12), set $a=1$, $b=2$, $y=-q^2$, and $x=-1$. Replace q with $-q$, divide both sides by $(1-q)(1-q^2)$ and use Slater's identity **69** to sum the resulting left side:

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2+2n}}{(q;q)_{2n+2}}=(-q^2,-q^{14},q^{16};q^{16})_\infty\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

The result follows after some slight manipulation.

The proof of (5.13) is similar, except we set $a = 1$, $b = 0$, $y = -1$, and $x = -1$, replace q with $-q$, and use Slater's identity **121**:

$$1 + \sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1} q^{n^2}}{(q;q)_{2n}} = \frac{(q^2, q^{14}, q^{16}; q^{16})_\infty (q^{12}, q^{20}; q^{32})_\infty}{(q;q)_\infty}.$$

□

6. Representation of False Theta Series as basic Hypergeometric Series

In this section we derive some new representations of the false theta series $\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}$ and $\sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1})$, as basic hypergeometric series.

On page 13 of the Lost Notebook [16] (see also [3, page 229]), Ramanujan recorded the following identity (amongst others in a similar vein):

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n (-1)^n q^{n^2+n}}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}. \tag{6.1}$$

On page 37 of the Lost Notebook, he recorded the identities

$$\begin{aligned} \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}) &= \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(-q;q)_{2n+1}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n}. \end{aligned} \tag{6.2}$$

The identity that follows from equating the left side to the second right side above also follows as a special case of a more general identity first stated by Rogers [18].

We use these identities in conjunction with (5.14) to prove the following.

Corollary 15.

$$1 - \sum_{n=0}^{\infty} \frac{(q;q^2)_n (-1)^n q^{n^2-n}}{(-1;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}. \tag{6.3}$$

$$\frac{2}{1+q} - \sum_{n=0}^{\infty} \frac{q^{2n^2+3n}}{(-q;q)_{2n+1}(1+q^{2n+3})} = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}). \tag{6.4}$$

$$\frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-1;q)_{n+2}} = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}). \tag{6.5}$$

Proof. For (6.3), set $a = b = 1$, $y = q$ and $x = -1$ in (5.14). Then divide both sides of the resulting identity by $1 + q$, so that the left side becomes the left side of (6.1). The result follows after re-indexing the resulting sum on the right side, together with a little manipulation.

For (6.4), replace x with x/y in (5.14) and let $y \to \infty$ to get

$$\sum_{n=0}^{\infty} \frac{(-x)^n q^{(a+1)n^2+(b-1)n}}{(q^2x;q^2)_n(-q^{a+b};q^{2a})_n} = (1-x)\left(1-q^{a-b}\sum_{n=1}^{\infty} \frac{(-x)^n q^{(a+1)n^2+(b-2a-1)n}}{(x;q^2)_n(-q^{a+b};q^{2a})_n}\right). \tag{6.6}$$

Then set $a=1$, $b=2$, $x=-1$, and divide both sides by $1+q$ so that the left side becomes the first right side of (6.2). The result again follows, upon re-indexing the sum on the right side.

To get (6.3), set $y=0$ in (5.14), then $a=b=1/2$ and $x=-1$, so the left side becomes the second right side in (6.2). The result likewise follows after re-indexing the resulting sum on the right side. □

References

[1] George E. Andrews, *Bailey's transform, lemma, chains and tree.* Special functions 2000: current perspective and future directions (Tempe, AZ), 1–22, NATO Sci. Ser. II Math. Phys. Chem., **30**, Kluwer Acad. Publ., Dordrecht, 2001.

[2] George Andrews; Alexander Berkovich. *The WP-Bailey tree and its implications.* J. London Math. Soc. (2) **66** (2002), no. 3, 529–549.

[3] G.E. Andrews and B.C. Berndt, *Ramanujan's Lost Notebook, Part 1*, Springer, New York (2005).

[4] W. N. Bailey, *Some Identities in Combinatory Analysis.* Proc. London Math. Soc. **49** (1947) 421–435.

[5] W. N. Bailey, *Identities of the Rogers-Ramanujan type.* Proc. London Math. Soc., **50** (1949) 1–10.

[6] Peter Borwein; Petr Lisoněk; Colin Percival. *Computational investigations of the Prouhet-Tarry-Escott problem.* Math. Comp. **72** (2003), no. 244, 2063–2070 (electronic).

[7] David Bressoud, *Some identities for terminating q-series.* Math. Proc. Cambridge Philos. Soc. **89** (1981), no. 2, 211–223.

[8] Jack Chernick; *Ideal Solutions of the Tarry-Escott Problem.* Amer. Math. Monthly **44** (1937), no. 10, 626–633.

[9] G. Gasper, *Summation, transformation, and expansion formulas for bibasic series*, Trans. Amer. Math. Soc., **312** (1989), 257-277.

[10] George Gasper; Mizan Rahman *An indefinite bibasic summation formula and some quadratic, cubic and quartic summation and trans-formation formulas.* Can. J. Math. **42** (1990), 1-27.

[11] Ira Gessel; Dennis Stanton. *Applications of q-Lagrange inversion to basic hypergeometric series.* Trans. Amer. Math. Soc. **277** (1983), no. 1, 173–201.

[12] Basil Gordon,*Some continued fractions of the Rogers-Ramanujan type.* Duke Math. J. **32** 1965 741–748.

[13] H. Göllnitz,*Partitionen mit Differenzenbedingungen.* J. reine angew. Math. **225** (1967) 154–190.

[14] Q. Liu; X. Ma *On the Characteristic Equation of Well-Poised Baily Chains* - To appear.

[15] Carlos J. Moreno; Samuel S. Wagstaff, Jr. *Sums of squares of integers.* Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2006. xii+354 pp.

[16] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[17] L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math.Soc. **25** (1894),pp. 318-343.

[18] L. J. Rogers, *On two theorems of combinatory analysis and some allied identities*, Proc. London Math. Soc. 16 (1917) 315–336.

[19] Chen Shuwen, *The Prouhet-Tarry-Escott Problem.* http://euler.free.fr/eslp/TarryPrb.htm

[20] U. B. Singh, *A note on a transformation of Bailey.* Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 177, 111–116.

[21] L. J. Slater, *A new proof of Rogers's transformations of infinite series.* Proc. London Math. Soc. (2) 53, (1951). 460–475.

[22] L. J. Slater, *Further identities of the Rogers-Ramanujan type*, Proc. London Math.Soc. **54** (1952) 147–167.

[23] V. P. Spiridonov, *An elliptic incarnation of the Bailey chain.* Int. Math. Res. Not. 2002, no. **37**, 1945–1977.

[24] M. V. Subbarao; A. Verma. *Some summations of q-series by telescoping.* Pacific J. Math. **191** (1999), no. 1, 173–182.

[25] S. O. Warnaar, *Extensions of the well-poised and elliptic well-poised Bailey lemma.* Indag. Math. (N.S.) **14** (2003), no. 3-4, 571–588.

Mathematics Department, Anderson Hall, West Chester University, West Chester, PA 19383

E-mail address: `jmclaughl@wcupa.edu`

Mathematics Department, Anderson Hall, West Chester University, West Chester, PA 19383

E-mail address: `pzimmer@wcupa.edu`