ASYMPTOTICS AND SEQUENTIAL CLOSURES OF CONTINUED FRACTIONS AND GENERALIZATIONS

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We would like to dedicate this paper to our mathematical father and grandfather, respectively, Basil Gordon.

Abstract. Given a sequence of complex square matrices, a_n, consider the sequence of their partial products, defined by $p_n = p_{n-1}a_n$. What can be said about the asymptotics as $n \to \infty$ of the sequence $f(p_n)$, where f is a continuous function? This paper addresses this question under the assumption that the matrices a_n are an l_1 perturbation of a sequence of matrices with bounded partial products. We chiefly apply the result to investigate the asymptotics of the approximants of continued fractions. In particular, when a continued fraction is l_1 limit 1-periodic of elliptic type, we show that the set of limits of its sequence of approximants have closures which are circles in $\mathbb{C}$, or are a finite set of points lying on a circle. More generally, similar results are found in the context of Banach algebras. The theory is also applied to (r, s)-matrix continued fractions, and recurrence sequences of Poincaré type.

1. Introduction

Consider the following recurrence:

$$x_{n+1} = \frac{3}{2} - \frac{1}{x_n}.$$

Taking $1/\infty$ to be 0 and vice versa, then regardless of the initial (real) value of this sequence, it is an interesting fact that the sequence is dense in $\mathbb{R}$. The proof is illuminating.

Take $x_0 = 3/2$ and view x_n as n'th approximant of the continued fraction:

$$3/2 - \frac{1}{3/2-}\,\frac{1}{3/2-}\,\frac{1}{3/2-}\,\frac{1}{3/2 - ...}. \tag{1.1}$$

Then, from the standard theorem on the recurrence for convergents of a continued fraction, the n'th numerator and denominator convergents of this

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continued fraction, A_n and B_n respectively, must both satisfy the linear recurrence relation

$$Y_n = \frac{3}{2}Y_{n-1} - Y_{n-2},$$

but with different initial conditions.

Now, the characteristic roots of this equation are $\alpha = 3/4 + i\sqrt{7}/4$, and $\beta = 3/4 - i\sqrt{7}/4$. Thus from the usual formula for solving linear recurrences, the exact formula for x_n is

$$x_n = \frac{A_n}{B_n} = \frac{a\alpha^n + b\beta^n}{c\alpha^n + d\beta^n} = \frac{a\lambda^n + b}{c\lambda^n + d},$$

where a, b, c, and d are some complex constants and $\lambda = \alpha/\beta$. Notice that λ is a number on the unit circle and is not a root of unity, so that λ^n is dense on the unit circle. The conclusion follows by noting that the linear fractional transformation

$$z \mapsto \frac{az + b}{cz + d}$$

is a homeomorphism from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$ and must take the unit circle to $\mathbb{R}$, since the values of the sequence x_n are real. Starting with other real x_0 just changes the constants in the transformation, so with a small modification the proof works for other real starting values.

After seeing this argument, one is tempted to write down the equality

$$\mathbb{R} = 3/2 - \frac{1}{3/2-}\,\frac{1}{3/2-}\,\frac{1}{3/2-}\,\frac{1}{3/2 - \ldots}.$$

This is true so long as one interprets the value of the continued fraction to be the set of limits of subsequences of its sequence of approximants. In this paper we generalize such equalities.

Another motivating example of our work is the following theorem, one of the oldest in the analytic theory of continued fractions [19, 31, 32]:

Theorem 1. *(Stern-Stolz,* [19, 31, 32]*) Let the sequence* $\{b_n\}$ *satisfy* $\sum |b_n| < \infty$. *Then*

$$b_0 + K_{n=1}^{\infty} \frac{1}{b_n}$$

diverges. In fact, for $p = 0, 1$,

$$\lim_{n\to\infty} P_{2n+p} = A_p \neq \infty, \qquad \lim_{n\to\infty} Q_{2n+p} = B_p \neq \infty,$$

and

$$A_1 B_0 - A_0 B_1 = 1.$$

The Stern-Stolz theorem shows that all continued fractions of the general form described in the theorem tend to two different limits, respectively A_0/B_0, and A_1/B_1. (These limits depend on the continued fraction.) Here and throughout we assume the limits for continued fractions are in $\widehat{\mathbb{C}}$. The motivation for this is that continued fractions can be viewed as the composition of linear fractional transformations and such functions have $\widehat{\mathbb{C}}$ as their natural domain and codomain.

Before leaving the Stern-Stolz theorem, we wish to remark that although the theorem is sometimes termed a "divergence theorem", this terminology is a bit misleading; the theorem actually shows that although the continued fractions of this form diverge, they do so by tending to two limits according to the parity of the approximant's index.

A special case of the Stern-Stolz theorem is a fact about the famous Rogers-Ramanujan continued fraction:

$$1+\frac{q}{1}\,+\,\frac{q^2}{1}\,+\,\frac{q^3}{1}\,+\,\frac{q^4}{1}\cdots. \tag{1.2}$$

The Stern-Stolz theorem gives that for $|q|>1$ the even and odd approximants of (1.2) tend to two limiting functions. To see this, observe that by the standard equivalence transformation for continued fractions, (1.2) is equal to

$$1+\frac{1}{1/q}\,+\,\frac{1}{1/q}\,+\,\frac{1}{1/q^2}\,+\,\frac{1}{1/q^2}\cdots\,+\,\frac{1}{1/q^n}\,+\,\frac{1}{1/q^n}\cdots.$$

The Stern-Stolz theorem, however does not apply to the following continued fraction given by Ramanujan:

$$\frac{-1}{1+q}\,+\,\frac{-1}{1+q^2}\,+\,\frac{-1}{1+q^3}\,+\cdots. \tag{1.3}$$

Recently in [1] Andrews, Berndt, *et al.* proved a claim made by Ramanujan in his lost notebook ([24], p.45) about (1.3). To describe Ramanujan's claim, we first need some notation. Throughout take $q\in\mathbb{C}$ with $|q|<1$. The following standard notation for q-products will also be employed:

$$(a)_0:=(a;q)_0:=1,\quad (a)_n:=(a;q)_n:=\prod_{k=0}^{n-1}(1-a\,q^k),\qquad \text{if } n\geq 1,$$

and

$$(a;q)_\infty:=\prod_{k=0}^{\infty}(1-a\,q^k),\qquad\qquad |q|<1.$$

Set $\omega=e^{2\pi i/3}$. Ramanujan's claim was that, for $|q|<1$,

$$\lim_{n\to\infty}\left(\frac{1}{1}\,-\,\frac{1}{1+q}\,-\,\frac{1}{1+q^2}\,-\cdots-\,\frac{1}{1+q^n+a}\right)=-\omega^2\left(\frac{\Omega-\omega^{n+1}}{\Omega-\omega^{n-1}}\right)\cdot\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}, \tag{1.4}$$

where

$$\Omega:=\frac{1-a\omega^2}{1-a\omega}\,\frac{(\omega^2q,q)_\infty}{(\omega q,q)_\infty}.$$

Ramanujan's notation is confusing, but what his claim means is that the limit exists as $n\to\infty$ in each of the three congruence classes modulo 3, and that the limit is given by the expression on the right side of (1.4). Also, the appearance of the variable a in this formula is a bit of a red herring; from elementary properties of continued fractions, one can derive the result for general a from the $a = 0$ case.

The continued fraction (1.1), the Stern-Stolz theorem, and (1.3) are, in fact, examples of the same phenomenon. We define this phenomenon and investigate its implications.

Now (1.1) is different from the other two examples in that it has subsequences of approximants tending to infinitely many limits. Nevertheless, all of the examples above, including (1.1), are special cases of a general result on continued fractions (Theorem 7 below). To deal with both of these situations we introduce the notion of the *sequential closure* of a sequence.

Define the *sequential closure* of the sequence in a topological space to be the set of limits of convergent subsequences.[1] To avoid confusion we designate the sequential closure of a sequence $\{s_n\}_{n\geq 1}$ by $s.c.(s_n)$.

In this paper we study sequential closures in the specific context of sequences of the form

$$f\left(\prod_{i=1}^{n} D_i\right),$$

where D_i are elements in a unital Banach algebra and f is a function with values in a metric space, often compact. Usually in this paper D_i is a sequence of complex matrices.

The main results of section 2 are Theorems 2, 3, and 4 which are the most general result of the paper. Theorem 2 is the most general and is stated in the setting of Banach algebras. In section 2 we also discuss recent results of Beardon, [3], which apply hyperbolic geometry to the analytic convergence theory of continued fractions. Some of the results of [3] are related to ours in as much as they deal with generalizing the Stern-Stolz theorem. Indeed, one of the conclusions of Theorem 2 is implied by one of the theorems from [3]. The principle difference is that the theorems of [3], which generalize the Stern-Stolz theorem, do not generalize the particular conclusion of the Stern-Stolz theorem that the continued fraction's even and odd approximants tend to two different limits, and instead focus on the fact of divergence. The approach of this present paper is to generalize the *convergence* of subsequences in the Stern-Stolz theorem.

A special case of Theorem 2 is Theorem 4 which is used to prove Theorem 7 which gives detailed information about the sequential closures of continued fractions. This result is studied in detail in sections 3 and 4. Section 5

[1]Thus, for example, the sequence $\{1, 1, 1, \dots\}$ has sequential closure $\{1\}$ although the set of limit (accumulation) points of the set of values of the sequence is empty. Note that in a survey paper describing some of the research in this paper, the authors previously used the phrase "limit set", unaware of the use of this phrase in the theories of discrete groups and dynamical systems. We thank Peter Loeb for the suggestion of the phrase "sequential closure".

and 6 use Theorem 4 to study (r, s)-matrix continued fractions, and linear recurrences of Poincaré type, respectively.

Section 3 focuses on limit periodic continued fractions of elliptic and loxodromic types. We discover a rich tapestry of results which weave together the sequential closure, modifications of the continued fraction, and the asymptotics of the approximants of a large class of continued fractions (including many which represent naturally occurring special functions). Those of elliptic type do not converge, but we find that their sequential closures are well behaved, computable, and that their approximants have nice asymptotics. This elliptic case has not previously been studied as far as we know.

We also address the statistics of the sequential closure. In particular, suppose a continued fraction (or matrix generalization) has an infinite sequential closure. Then which points in the set have the "most" approximants tending to them, and which have the "fewest"? Thus for example, the approximants of the continued fraction for $\mathbb{R}$ above hovers most frequently around which real value(s)? These question are answered simply by considering the geometry of the relevant linear fractional transformation.

Section 4 studies a non-trivial example of the theory. The section concerns a particular continued fraction with three parameters which generalizes not only the "3/2 continued fraction" above, but also the continued fraction (1.4). An example of this theorem is a perturbation of the "3/2 continued fraction", specifically, the sequential closure of the continued fraction:

$$3/2 - \frac{1}{q + 3/2 -}\,\frac{1}{q^2 + 3/2 -}\,\frac{1}{q^3 + 3/2 -}\,\frac{1}{q^4 + 3/2\ - \cdots}, \tag{1.5}$$

where $|q| < 1$ is complex, can be described exactly. In fact, this sequential closure is a circle on the Riemann sphere. (Thus as a consequence, when $|q| < 1$ and q is real, (1.5) always has sequential closure $\mathbb{R}$.) Viewing this circle as a linear fractional transformation of the unit circle $\{z \in \mathbb{C} : |z| = 1\}$,

$$z \mapsto \frac{az + b}{cz + d},$$

it transpires that the parameters a, b, c, and d are basic hypergeometric functions.

More generally, in (1.5) if the numbers 1 and 3/2 are changed so that the limiting recurrence for the convergents of the continued fraction have distinct characteristic roots that are on the unit circle, there is a coherent formula, in terms of basic hypergeometric functions, for the sequential closure regardless of the nature of the roots on the unit circle. Indeed, cases in which the characteristic roots are roots of unity lead to cases where the continued fraction has a finite set of limits. Theorem 10 is the general result.

Remarks:

(i) All sequential closure equalities in this paper arise from the situation

$$\lim_{n\to\infty} d(s_n, t_n) = 0$$

in some metric space (X, d). Accordingly, it makes sense to define the equivalence relation $\sim$ on sequences in X by $\{s_n\} \sim \{t_n\} \iff \lim_{n\to\infty} d(s_n, t_n) = 0$. In this situation we refer to sequences $\{s_n\}$ and $\{t_n\}$ as being asymptotic to each other. Abusing notation, we often write $s_n \sim t_n$ in place of $\{s_n\} \sim \{t_n\}$. More generally, we frequently write sequences without braces when it is clear from context that we are speaking of a sequence, and not the nth term. Note that the statements $\lim_{n\to\infty} x_n = L$ and $x_n \sim L$ are equivalent. In this paper, the general theorems are given in the case where the metric space is a unital Banach algebra; the theorems are then applied to spaces of matrices.

(ii) It is a fact from general topology that given a compact topological space X and a Hausdorff space Y, then any continuous bijection $g : X \to Y$ must be a homeomorphism and g and its inverse must both be uniformly continuous. Under these assumptions an immediate consequence for sequential closures is: *If $\{s_n\}_{n\geq 1}$ is a sequence with values in X, then $s.c.(g(s_n)) = g(s.c.(s_n))$.*

(iii) Another basic fact is that *If $\{s_n\}$ and $\{t_n\}$ are two sequences in some metric space satisfying $s_n \sim t_n$, then $s.c.(s_n) = s.c.(t_n)$.* Additionally, if f is some uniformly continuous function, then the following sequence of implications holds:

$$s_n \sim t_n \implies f(s_n) \sim f(t_n) \implies s.c.(f(s_n)) = s.c.(f(t_n)).$$

2. Asymptotics and sequential closures of infinite products in unital Banach algebras

The classic theorem on the convergence of infinite products of matrices seems to have been given first by Wedderburn [36, 37]. Wedderburn's theorem is maybe not as well known as it deserves to be, perhaps because Wedderburn does not state it explicitly as a theorem, but rather gives inequalities from which the convergence of infinite matrix products can be deduced under an l_1 assumption. Wedderburn also provides the key inequality for establishing the invertibility of the limit, but does not discuss this important application of his inequality. It is not hard to see that Wedderburn's equations hold in any unital Banach algebra. Because of these factors, we provide both the statement of the theorem as well as its proof in the setting of a unital Banach algebra. We will immediately apply the theorem to obtain our most general result, which gives asymptotics for oscillatory divergent infinite products in Banach algebras. This theorem is then applied to the Banach algebra $M_d(\mathbb{C})$ of $d \times d$ matrices of complex numbers topologised using the l_∞ norm, denoted by $||\cdot||$.

For any unital Banach algebra, let I denote the identity. When we use product notation for elements of a Banach algebra, or for matrices, the product is taken from left to right; thus

$$\prod_{i=1}^{n} A_i := A_1 A_2 \cdots A_n.$$

Theorems with products taken in the opposite order follow from the theorems below by taking the products in the reverse order throughout the statements and proofs.

Proposition 1. *(Wedderburn [36, 37]) Let the sequence A_i consist of elements of a unital Banach algebra $\mathbf{U}$ for $i \geq 1$. Then $\sum_{i\geq 1} ||A_i|| < \infty$ implies that $\prod_{i\geq 1}(I + A_i)$ converges in $\mathbf{U}$. Moreover, all the elements of the sequence $I + A_i$ are invertible if and only if the limit $\prod_{i\geq 1}(I + A_i)$ is invertible.*

The following corollary provides a convenient estimate of the convergence rate of the product.

Corollary 1. *Under the conditions of Proposition 1, let $L = \prod_{i\geq 1}(I + A_i)$ and $P_m = \prod_{i=1}^{m}(I + A_i)$. Then*

$$||L - P_m|| \leq e^{\sum_{i\geq 1} ||A_i||} - e^{\sum_{1\leq i\leq m} ||A_i||} = O\left(\sum_{i>m} ||A_i||\right). \tag{2.1}$$

Proof of Proposition (Wedderburn). Put

$$P_m = (I + A_1)(I + A_2) \cdots (I + A_m),$$

and

$$Q_m = (1 + ||A_1||)(1 + ||A_2||) \cdots (1 + ||A_m||).$$

Expanding the product for P_m gives

$$P_m = I + \sum_{1\leq n_1\leq m} A_{n_1} + \sum_{1\leq n_1<n_2\leq m} A_{n_1}A_{n_2} + \sum_{1\leq n_1<n_2<n_3\leq m} A_{n_1}A_{n_2}A_{n_3} + \cdots + A_1 A_2 \cdots A_m. \tag{2.2}$$

Similarly,

$$Q_m = 1 + \sum_{1\leq n_1\leq m} ||A_{n_1}|| + \sum_{1\leq n_1<n_2\leq m} ||A_{n_1}||\,||A_{n_2}|| + \sum_{1\leq n_1<n_2<n_3\leq m} ||A_{n_1}||\,||A_{n_2}||\,||A_{n_3}|| + \cdots + ||A_1||\,||A_2|| \cdots ||A_m||.$$

Thus for $m \geq k$,

$$||P_m - P_k|| \leq Q_m - Q_k, \tag{2.3}$$

and

$$||P_m - I|| \leq Q_m - 1 < e^{\sum_{n\geq 1} ||A_n||} - 1. \tag{2.4}$$

From the standard condition for the convergence of infinite products of complex numbers, the convergence of $\sum_{n\geq 1}||A_n||$ implies the convergence of $\prod_{i\geq 1}(1+||A_i||)$, and this implies that the sequence Q_n is Cauchy. Thus by (2.3), P_m is also Cauchy, and so $\prod_{i\geq 1}(I+A_i)$ exists.

Recall that an element x in a Banach algebra is invertible if $||x-I||<1$. For $\prod_{i\geq 1}(I+A_i)$ to be invertible, it is obviously necessary that the elements of the sequence $I+A_i$ be invertible. We show that this is sufficient. Since $\sum_{i\geq 1}||A_i||<\infty$, there exists $j\in\mathbb{Z}^+$ such that $\sum_{n>j}||A_n||<\log(2)$. Then (2.4) gives that

$$||(I+A_{j+1})\cdots(I+A_{j+m})-I||<e^{\sum_{n>j}||A_n||}-1.$$

Letting $m\to\infty$ yields

$$\lim_{m\to\infty}||(I+A_{j+1})\cdots(I+A_{j+m})-I||\leq e^{\sum_{n>j}||A_n||}-1<e^{\log(2)}-1=1.$$

Hence $\lim_{m\to\infty}(I+A_{j+1})\cdots(I+A_{j+m})$ is invertible. Multiplying this on the left by the invertible elements $I+A_i$, $1\leq i\leq j$ gives the conclusion. $\square$

Proof of Corollary. From Proposition 1,

$$\begin{aligned}||L-P_m||&=\left\|\prod_{i\geq 1}(1+A_i)-\prod_{1\leq i\leq m}(1+A_i)\right\|\\&\leq\left\|\prod_{1\leq i\leq m}(1+A_i)\right\|\left\|\prod_{i>m}(1+A_i)-I\right\|\\&\leq e^{\sum_{1\leq i\leq m}||A_i||}(e^{\sum_{i>m}||A_i||}-1)=O\left(\sum_{i>m}||A_i||\right).\end{aligned}$$

$\square$

There have been a number of theorems more recently on the convergence of matrix products, see [2, 3, 5, 6, 9, 12, 27, 33, 34]. Closely related to Wedderburn's theorem are Theorems 3.7 and 3.8 of [3], originally given in [10], which gives essentially the same result, restricted to $SL_2(\mathbf{C})$.

Our focus here is on cases of divergence and our results concern finding asymptotics for the nth partial products. These in turn can be used to describe the sequential closures.

We set some further conventions and fix notation. Let G be a metric space, typically a subset of $\widehat{\mathbb{C}}^g$, where $\widehat{\mathbb{C}}$ is the Riemann sphere and g is some integer $g\geq 1$. Here $\widehat{\mathbb{C}}$ is topologised with the chordal metric and the corresponding product metric is employed for $\widehat{\mathbb{C}}^g$. (This is defined by taking the maximum of the metrics of all the corresponding elements in two g-tuples.) Let f be a continuous function from a compact subset (to be specified) of a unital Banach algebra $\mathbf{U}$, (usually $M_d(\mathbb{C})$) to G. Typically we do not distinguish different norms, the correct one being supplied from context. The topological closure of a set S is denoted by $\overline{S}$.

Our first theorem is a perturbation result giving the asymptotics of divergent infinite products in a unital Banach algebra. Although we will only use a special case of this result, we believe the general result is of sufficient interest to warrant inclusion, especially since the proof of the general result requires no additional work. We denote elements of the Banach algebra by capitol letters to suggest matrices, which is the case to which the result will be applied.

Theorem 2. *Suppose $\{M_i\}$ and $\{D_i\}$ are sequences in a unital Banach algebra $\mathbf{U}$ such that the two sequences (for $\epsilon = \pm 1$)*

$$\left\| \left(\prod_{i=1}^{n} M_i \right)^{\epsilon} \right\| \tag{2.5}$$

are bounded and $\{D_i - M_i\} \in l_1(\mathbf{U})$, that is,

$$\sum_{i \geq 1} \|D_i - M_i\| < \infty. \tag{2.6}$$

Let $\varepsilon_n = \sum_{i>n} ||D_i - M_i||$. Then

$$F := \lim_{n \to \infty} \left(\prod_{i=1}^{n} D_i \right) \left(\prod_{i=1}^{n} M_i \right)^{-1} \tag{2.7}$$

exists and F is invertible if and only if D_i is invertible for all $i \geq 1$. Also,

$$\left\| F - \left(\prod_{i=1}^{n} D_i \right) \left(\prod_{i=1}^{n} M_i \right)^{-1} \right\| = O(\varepsilon_n). \tag{2.8}$$

As sequences

$$\prod_{i=1}^{n} D_i \sim F \prod_{i=1}^{n} M_i, \tag{2.9}$$

and moreover

$$\left\| \prod_{i=1}^{n} D_i - F \prod_{i=1}^{n} M_i \right\| = O(\varepsilon_n). \tag{2.10}$$

More generally, let f be a continuous function from the domain

$$\overline{\left\{ F \prod_{i=1}^{n} M_i : n \geq h \right\}} \cup \bigcup_{n \geq h} \left\{ \prod_{i=1}^{n} D_i \right\},$$

for some integer $h \geq 1$, into a metric space G. Then the domain of f is compact in $\mathbf{U}$ and $f(\prod_{i=1}^{n} D_i) \sim f(F \prod_{i=1}^{n} M_i)$. Finally

$$s.c. \left(\prod_{i=1}^{n} D_i \right) = s.c. \left(F \prod_{i=1}^{n} M_i \right), \tag{2.11}$$

and

$$s.c.\left(f\left(\prod_{i=1}^{n} D_i\right)\right) = s.c.\left(f\left(F\prod_{i=1}^{n} M_i\right)\right). \tag{2.12}$$

We do not assume compactness of G so it is possible that the equalities in the theorem are between empty sets. When G is compact these sets are clearly non-trivial. Note that the conditions of the theorem imply that all the elements M_i are invertible. When $M_i = I$ for $i \geq 1$, the first conclusion of the theorem reduces to Wedderburn's theorem, Proposition 1.

An interesting special case of Theorem 2 is when the elements M_i are unitary matrices. The following matrix norm will be used:

$$||M|| = \left(\sum_{1\leq i,j\leq d} |m_{i,j}|\right)^{1/2}.$$

It is clear that $||M|| = \sqrt{d}$ when M is a $d \times d$ unitary matrix (for then $||M||^2 = \operatorname{tr}(M\overline{M}^{\mathrm{T}}) = \operatorname{tr}(I) = d$), and thus the hypothesis on the sequence M_i is satisfied. More generally, one can assume that the sequence of matrices $\{M_i\}$ are elements of some subgroup of $GL_d(\mathbb{C})$ that is conjugate to the unitary group. This case is important enough that we distinguish it in the following theorem.

Theorem 3. *Let $\{M_i\}$ be a sequence of elements of a subgroup of $GL_d(\mathbb{C})$ that is conjugate to the unitary group. Then, if $\{D_i\}$ is a sequence $GL_d(\mathbb{C})$ and $\{D_i - M_i\} \in l_1$, all of the conclusions of Theorem 2 hold.*

The special case of Theorem 2 that will be applied in the next section is $\mathbf{U} = M_d(\mathbb{C})$, $M_i = M$, where M be a diagonalizable complex matrix with eigenvalues on the unit circle. Since M is diagonalizable, put $M = CEC^{-1}$. Then $M^k = CE^kC^{-1}$, and so it follows that $\|M^k\| \leq \|C\| \cdot \|C^{-1}\|$ and $\|M^k\|$ is bounded for $k \in \mathbb{Z}$. Thus the boundedness hypothesis is satisfied and Theorem 2 simplifies to the following.

Theorem 4. *Under the above conditions,*

$$F = \lim_{n\to\infty}\left(\prod_{i=1}^{n} D_i\right) M^{-n}$$

exists in $M_d(\mathbb{C})$ and $\det(F) \neq 0$. Moreover, $||F - (\prod_{i=1}^{n} D_i)M^{-n}|| = O(\varepsilon_n)$, and

(i) $|| \prod_{i=1}^{n} D_i - FM^n|| = O(\varepsilon_n)$. Thus $s.c.(\prod_{i=1}^{n} D_i) = s.c.(FM^n)$.

(ii) Let f be a continuous function from the domain

$$\overline{\{FM^n : n \geq h\}} \cup \bigcup_{n\geq h}\left\{\prod_{i=1}^{n} D_i\right\},$$

for some integer $h \geq 1$, into a metric space G. Then the domain of f is compact and $f(\prod_{i=1}^{n} D_i) \sim f(FM^n)$. Hence $s.c.(f(\prod_{i=1}^{n} D_i)) = s.c.(f(FM^n))$.

Note that because M is diagonalizable, this theorem enables one to determine the exact structure of the sequential closure using Pontryagin duality.

A special case of Theorem 4 is Proposition 1 of [7] in which the eigenvalues are roots of unity. It considers asymptotics, but not the limit set. This special case is roughly equivalent to Theorem 1.1 of [28].

Proof of Theorem 2. Observe that

$$\begin{aligned}\left(\prod_{i=1}^{n} D_i\right)\left(\prod_{i=1}^{n} M_i\right)^{-1} &= \prod_{i=1}^{n}\left(\left(\prod_{j=1}^{i-1} M_j\right) D_i \prod_{j=0}^{i-1} M_{i-j}^{-1}\right)\\ &= \prod_{i=1}^{n}\left(I + \left(\prod_{j=1}^{i-1} M_j\right) D_i \prod_{j=0}^{i-1} M_{i-j}^{-1} - \left(\prod_{j=1}^{i-1} M_j\right) M_i \prod_{j=0}^{i-1} M_{i-j}^{-1}\right)\\ &= \prod_{i=1}^{n}\left(I + \left(\prod_{j=1}^{i-1} M_j\right)(D_i - M_i)\prod_{j=0}^{i-1} M_{i-j}^{-1}\right)\\ &= \prod_{i=1}^{n}(I + A_i),\end{aligned}$$

where

$$A_i := \left(\prod_{j=1}^{i-1} M_j\right)(D_i - M_i)\prod_{j=0}^{i-1} M_{i-j}^{-1}.$$

Hence

$$\|A_i\| \le \left\|\prod_{j=1}^{i-1} M_j\right\| \cdot \|D_i - M_i\| \cdot \left\|\prod_{j=0}^{i-1} M_{i-j}^{-1}\right\| \le C\|D_i - M_i\|,$$

for some real absolute bound C. The second inequality followed from the boundedness assumption on the sequences (2.5). By (2.6) it follows that $\sum_{i\ge 0}\|A_i\| < \infty$, and so by Proposition 1, it follows that F exists and is invertible when the D_i are invertible for $i \ge 1$. Thus we have proved that

$$\lim_{n\to\infty}\left\|F - \left(\prod_{i=1}^{n} D_i\right)\left(\prod_{i=1}^{n} M_i\right)^{-1}\right\| = 0. \tag{2.13}$$

Then again from the boundedness of the sequences in (2.5),

$$\lim_{n\to\infty}\left\|F\prod_{i=1}^{n} M_i - \prod_{i=1}^{n} D_i\right\| = 0. \tag{2.14}$$

That is,

$$\prod_{i=1}^{n} D_i \sim F\prod_{i=1}^{n} M_i.$$

Using this and the boundedness of the sequences in (2.5) gives that the domain of f is compact. Thus f is not only continuous, but is uniformly continuous. This uniform continuity and (2.14) give

$$\lim_{n\to\infty} \left\| f\left(F\prod_{i=1}^{n} M_i\right) - f\left(\prod_{i=1}^{n} D_i\right)\right\| = 0,$$

and so

$$f\left(F\prod_{i=1}^{n} M_i\right) \sim f\left(\prod_{i=1}^{n} D_i\right).$$

The sequential closure equalities in the theorem follow from the third remark in the introduction and the error estimates follow from Corollary 1 and the boundedness assumption. □

We conclude this section by comparing these results to some of those from the recent paper [3], which mainly focuses on applying the hyperbolic geometry of Möbius maps to the convergence theory of continued fractions with complex elements. Consider the following two results from [3] that are closely related the results of this section:

Theorem 5 (Theorem 4.2 of [3])**.** *Suppose that G is a topological group whose topology is derived from a right-invariant metric σ_0, and that (G, σ_0) is complete. Let $f_1, f_2, \dots$ be any sequence of elements of G. Then, for each k, there is a neighborhood $\mathcal{N}_k$ of f_k such that if, for all j, $g_j \in \mathcal{N}_j$, then $(g_1 \cdots g_n)(f_1 \cdots f_n)^{-1}$ converges to some element h of G.*

The above theorem shares some of the structure of Theorem 2. In particular it gives the existence of a limit similar to the limit F in Theorem 2. The hypotheses are quite different, however, and asymptotics are not given in Theorem 5. Also, sizes of the neighborhoods are not provided.

For the following corollary, some definitions involving hyperbolic geometry are useful. A *Möbius map* acting on $\widehat{\mathbb{R}}^N$ is a finite composition of maps each of which is an inversion or reflection in some $N-1$-dimensional hyperplane or hypersphere in $\widehat{\mathbb{R}}^N$. The *Möbius group* acting on $\widehat{\mathbb{R}}^N$ is the group generated by these inversions or reflections. The *conformal Möbius group*, denoted $\mathcal{M}_N$ is the subgroup of those maps that are orientation preserving which means that they can be expressed as the composition of an even number of such inversions. See [3, 4].

Corollary 2 (Corollary 4.3 of [3])**.** *Let $f_1, f_2, \dots$ be any sequence of Möbius maps. Then, for each k, there is a neighborhood $\mathcal{N}_k$ of f_k such that if $g_j \in \mathcal{N}_j$, $j = 1, 2, \dots$, then there is some Möbius map h such that for all z, $\sigma(g_1 \cdots g_n(z), hf_1 \cdots f_n)) \to 0$ as $n \to \infty$. In particular, for each point z, $\lim_n g_1 \cdots g_n(z)$ exists if and only if $\lim_n f_1 \cdots f_n(z)$ exists.*

The differences with our theorem are that the setting in Theorem 2 is more general and the sizes of the neighborhoods are not given in Corollary

2. However, in the case of complex Möbius maps, in [3] it is shown that the neighborhoods $\mathcal{N}_k$ can be taken to be the set of Möbius maps g that satisfy

$$||g - f_k|| < \frac{1}{2^{k+2}||f_1||^2 \cdots ||f_{k-1}||^2||f_k||}. \tag{2.15}$$

Here the norms are of the matrix representations of the Möbius maps f_i and g.

Comparing this with Theorem 2, it can be seen that for the case of complex Möbius maps, unless enough of the norms $||f_i||$ are small, one expects our condition $\{D_i - M_i\} \in l_1$ to be weaker in general, and thus our result to be stronger. Note that Theorem 2 also gives information about the sequential closure as well as asymptotics with error terms. Information about the sequential closure is implicit, however, in Corollary 4.3 of [3] above.

There is another theorem in [3] which is also related to Theorem 2. In fact, it is a generalization of the Stern-Stolz theorem presented in the introduction. Before stating the theorem, a couple definitions concerning the hyperboloid model of hyperbolic space are required.

For x and y in $\mathbb{R}^{N+1}$, let

$$q(x, y) = x_1y_1 + x_2y_2 + \cdots + x_Ny_N - x_{N+1}y_{N+1},$$

and

$$\mathcal{H}_N = \{x \in \mathbb{R}^{N+1} : q(x, x) = 1,\, x_{N+1} > 0\}.$$

$\mathcal{H}_N$ is one branch of a hyperboloid of two sheets. It can be shown that $\mathcal{H}_N$ can be endowed with a hyperbolic metric and that the matrix group $O^+(N + 1, 1)$ which preserve q as well as th condition $x_{N+1} > 0$ act as isometries on this space. Let g be a Möbius map which acts on $\mathbb{R}^N$, and hence by the Poincaré extension, on $\mathbb{H}^{N+1}$. Suppose then that g corresponds to the $(N+2)\times(N+2)$ matrix A which acts on $\mathcal{H}_{N+2}$. In [3] the following beautiful generalization of the Stern-Stolz theorem is given:

Theorem 6 ("The General Stern-Stolz Theorem" [3])**.** *Suppose that g_1, g_2 ... are Möbius maps in $\mathcal{M}_N$, and that g_n is represented by the $(N + 2) \times (N + 2)$ matrix A_n as above. If*

$$\sum_{n=1}^{\infty} \sqrt{||A_n||^2 - ||I||^2} \tag{2.16}$$

converges, then the sequence $g_1 \cdots g_n$ is strongly divergent.

Consider the $N = 0$ case. Then, this theorem should be compared with the case of Theorem 3 in which H is unitary, and the matrices M_i represent Möbius maps. In Theorem 6, (2.16) is exactly the condition required for $\sum_n \rho(\mathbf{j}, g_n(\mathbf{j}))$ to be bounded in $\mathbb{H}$. (Here ρ is the hyperbolic metric on $\mathbb{H}$, where $\mathbb{H} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$, and $\mathbf{j} = (0, 0, 1)$.) Next, (2.16) is sufficient to guarantee that $\rho(\mathbf{j}, g_1 \cdots g_n(\mathbf{j}))$ is finite, and thus the orbits of the product $g_1 \cdots g_n$ never leave $\mathbb{H}$. This later condition is what is meant by "strong divergence". Now the Möbius maps that fix $\mathbf{j}$ are the unitary maps and $g(\mathbf{j}) = \mathbf{j}$ if and only if $||g||^2 = 2$. The condition (2.16) can thus be interpreted as saying that the elements g_n approach some sequence of unitary elements sufficiently rapidly. This is roughly the same as the condition on the sequence $\{D_i\}$ in Theorem 3 when H is unitary. Of course the conclusion of the theorems go in different directions.

In the next section we apply the $d = 2$ case of Theorem 4 to get detailed information about the sequential closures of continued fractions.

3. Limit 1-periodic continued fractions of elliptic type

We begin by reviewing the correspondence between 2×2 matrices and continued fractions. First recall that a finite continued fraction is a rational function of the form:

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots \cfrac{a_{n-1}}{b_{n-1} + \cfrac{a_n}{b_n}}}}.$$

For easier reading continued fractions are usually typeset as:

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n}.$$

The correspondence between continued fractions and matrices is best understood by first remembering the correspondence between compositions of linear fractional transformations and products of 2×2 matrices, and then noting that the composition of linear fractional transformations can be written as a continued fraction. To see the later, observe that for a general linear fractional transformation (avoiding cases such as $c = 0$):

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{\left(\frac{bc-ad}{c^2}\right)}{\frac{d}{c} + z},$$

Thus, generically, any composition of a finite number of non-trivial linear fractional transformations can be written as a finite continued fraction. But to generate a continued fraction, one does not need to work with such general linear fractional transformations. For example, working with transformations of the form

$$\left(\frac{a_i}{b_i + z^{-1}}\right)^{-1} = \frac{bz+1}{az}$$

leads to the correspondence between matrices and continued fractions that will be used below:

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_1 & 1 \\ a_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_n & 1 \\ a_n & 0 \end{pmatrix}, \tag{3.1}$$

where

$$\frac{P_n}{Q_n} = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n}.$$

Here P_n and Q_n are the numerator and denominator polynomials (called *convergents*) in the variables a_i and b_i obtained by simplifying the rational function that is the finite continued fraction. Their ratio, P_n/Q_n, is called the *nth approximant* of the continued fraction. From (3.1) one reads off immediately the fundamental recurrences for the convergents P_n and Q_n:

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} P_{n-1} & P_{n-2} \\ Q_{n-1} & Q_{n-2} \end{pmatrix} \begin{pmatrix} b_n & 1 \\ a_n & 0 \end{pmatrix}. \tag{3.2}$$

Taking the determinant on both sides of (3.1) gives at once the *determinant formula* for the convergents of a continued fraction:

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} a_1 a_2 \cdots a_n. \tag{3.3}$$

An infinite continued fraction

$$K_{n=1}^{\infty} \frac{a_n}{b_n} := \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \tag{3.4}$$

is said to converge in $\mathbb{C}$ (respectively in $\widehat{\mathbb{C}}$) if

$$\lim_{n\to\infty} \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n}$$

exists in $\mathbb{C}$ (respectively in $\widehat{\mathbb{C}}$). Let $\{\omega_n\}$ be a sequence of complex numbers. If

$$\lim_{n\to\infty} \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n + \omega_n}$$

exist, then this limit is called the *modified limit of* $K_{n=1}^{\infty} a_n/b_n$ *with respect to the sequence* $\{\omega_n\}$. Detailed discussions of modified continued fractions as well as further pointers to the literature are given in [19]. Note that by (3.1) and (3.2),

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n + \omega_n} = \frac{P_n + \omega_n P_{n-1}}{Q_n + \omega_n Q_{n-1}}. \tag{3.5}$$

In the following theorem, the sequential closure of the sequence of approximants of a general class of continued fractions is computed. It transpires that the sequential closure is a circle (or a finite subset of a circle) on the Riemann sphere.

The following theorem studies the continued fraction

$$\frac{-\alpha\beta + q_1}{\alpha + \beta + p_1} + \frac{-\alpha\beta + q_2}{\alpha + \beta + p_2} + \cdots + \frac{-\alpha\beta + q_n}{\alpha + \beta + p_n}, \tag{3.6}$$

where the sequences p_n and q_n approach 0 in l_1 and the constants $\alpha \neq \beta$ are points in the complex plane. Specifically assume that

$$\sum_{n=1}^{\infty} |p_n| < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} |q_n| < \infty. \tag{3.7}$$

Let

$$\varepsilon_n := \max\left(\sum_{i>n}|p_i|, \sum_{i>n}|q_i|\right),$$

and put

$$f_n(w) := \frac{-\alpha\beta + q_1}{\alpha+\beta+p_1} + \frac{-\alpha\beta + q_2}{\alpha+\beta+p_2} + \cdots + \frac{-\alpha\beta + q_n}{\alpha+\beta+p_n+w},$$

so that $f_n := P_n/Q_n = f_n(0)$ is the sequence of approximants of the continued fraction (3.6).

We follow the common convention in analysis of denoting the group of points on the unit circle by $\mathbb{T}$, (and also by $\mathbb{T}_\infty$), and its subgroup of roots of unity of order m, m finite, by $\mathbb{T}_m$. (Note: $\mathbb{T}_\infty$ often denotes the group of all roots of unity; here it denotes the whole circle group.)

Theorem 7. *Let $\{p_n\}_{n\geq 1}$, $\{q_n\}_{n\geq 1}$ be complex sequences satisfying* (3.7). *Let α and β satisfy $|\alpha| = |\beta| = 1$, $\alpha \neq \beta$ with the order of $\lambda = \alpha/\beta$ in $\mathbb{T}$ being m (where m may be infinite). Assume that $q_n \neq \alpha\beta$ for any $n \geq 1$. The following asymptotics for the convergents P_n and Q_n hold as $n \to \infty$:*

$$\left|P_n - \frac{a\alpha^n + b\beta^n}{\alpha - \beta}\right| = O(\varepsilon_n) \quad \text{and} \quad \left|Q_n - \frac{c\alpha^n + d\beta^n}{\alpha - \beta}\right| = O(\varepsilon_n). \tag{3.8}$$

Also

$$f_n \sim h(\lambda^{n+1}) \quad \text{where} \quad h(z) = \frac{az+b}{cz+d}. \tag{3.9}$$

Moreover

$$s.c.\,(f_n) = h(\mathbb{T}_m) = \frac{a\mathbb{T}_m + b}{c\mathbb{T}_m + d}, \tag{3.10}$$

with the constants $a, b, c, d \in \mathbb{C}$ given by the (existent) limits

$$\begin{aligned} a &= \lim_{n\to\infty} \alpha^{-n}(P_n - \beta P_{n-1}), \\ b &= -\lim_{n\to\infty} \beta^{-n}(P_n - \alpha P_{n-1}), \\ c &= \lim_{n\to\infty} \alpha^{-n}(Q_n - \beta Q_{n-1}), \\ d &= -\lim_{n\to\infty} \beta^{-n}(Q_n - \alpha Q_{n-1}). \end{aligned} \tag{3.11}$$

Also,

$$\det(h) = ad - bc = (\beta - \alpha)\prod_{n=1}^{\infty}\left(1 - \frac{q_n}{\alpha\beta}\right) \neq 0. \tag{3.12}$$

Finally, if either $|c| \neq |d|$, or for $0 \leq n < m$, $c\lambda^n + d \neq 0$ when $|c| = |d|$ and λ is a root of unity, then as $n \to \infty$,

$$\left|f_n - h(\lambda^{n+1})\right| = O(\varepsilon_n). \tag{3.13}$$

This theorem is foundational for what follows. We give two corollaries before the proof. Further results follow the proofs. The next corollary gives enough information for specifically identifying the linear fractional transformation h in the theorem in terms of modifications of the original continued fraction. The succeeding corollary makes that identification.

Corollary 3. *Under the conditions of the theorem the following identities involving modified versions of* (3.6) *hold in* $\widehat{\mathbb{C}}$*:*

(3.14)

$$h(\infty) = \frac{a}{c} = \lim_{n\to\infty} f_n(-\beta) = \lim_{n\to\infty} \frac{-\alpha\beta + q_1}{\alpha+\beta+p_1} + \frac{-\alpha\beta + q_2}{\alpha+\beta+p_2} + \cdots + \frac{-\alpha\beta + q_{n-1}}{\alpha+\beta+p_{n-1}} + \frac{-\alpha\beta + q_n}{\alpha+p_n};$$

(3.15)

$$h(0) = \frac{b}{d} = \lim_{n\to\infty} f_n(-\alpha) = \lim_{n\to\infty} \frac{-\alpha\beta + q_1}{\alpha+\beta+p_1} + \frac{-\alpha\beta + q_2}{\alpha+\beta+p_2} + \cdots + \frac{-\alpha\beta + q_{n-1}}{\alpha+\beta+p_{n-1}} + \frac{-\alpha\beta + q_n}{\beta+p_n};$$

and for $k \in \mathbb{Z}$*, we have*

$$h(\lambda^{k+1}) = \frac{a\lambda^{k+1}+b}{c\lambda^{k+1}+d} = \lim_{n\to\infty} f_n(\omega_{n-k})$$

$$= \lim_{n\to\infty} \frac{-\alpha\beta + q_1}{\alpha+\beta+p_1} + \frac{-\alpha\beta + q_2}{\alpha+\beta+p_2} + \cdots + \frac{-\alpha\beta + q_n}{\alpha+\beta+p_n+\omega_{n-k}}, \tag{3.16}$$

where

$$\omega_n = -\frac{\alpha^n - \beta^n}{\alpha^{n-1}-\beta^{n-1}} \in \widehat{\mathbb{C}}, \qquad n \in \mathbb{Z}.$$

The following corollary gives (up to a factor of ± 1) the numbers a, b, c, and d in terms of the (convergent) modified continued fractions given in Corollary 3.

Corollary 4. *The linear fractional transformation $h(z)$ defined in Theorem 7 has the following expression*

$$h(z) = \frac{A(C-B)z + B(A-C)}{(C-B)z + A - C},$$

where $A = h(\infty)$, $B = h(0)$, and $C = h(1)$. Moreover, the constants a, b, c, and d in the theorem have the following formulas

$$a = sA(C-B), \quad b = sB(A-C), \quad c = s(C-B), \quad d = s(A-C),$$

where

$$s = \pm\sqrt{\frac{(\beta-\alpha)\prod_{n=1}^{\infty}\left(1 - \frac{q_n}{\alpha\beta}\right)}{(A-B)(C-A)(B-C)}}.$$

Some remarks before the proofs:

(i) It is interesting to note that the sequence of modifications of (3.6) occurring in (3.16) converge exactly to the sequence $h(\lambda^{n+1})$ which is asymptotic to the approximants f_n of (3.6).

(ii) Dividing through the numerator and denominator of the definition of ω_n by β^{n-1} gives that the sequence ω_n occurring in (3.16) is either a discrete or a dense set of points on the line

$$\frac{-\alpha\mathbb{T}+\beta}{\mathbb{T}+1},$$

according to whether λ is a root of unity or not. Observe that $-\omega_{n+2}$ is the nth approximant of the continued fraction

$$\alpha+\beta+\frac{-\alpha\beta}{\alpha+\beta}+\frac{-\alpha\beta}{\alpha+\beta}+\cdots\frac{-\alpha\beta}{\alpha+\beta},$$

which, except for the initial $\alpha+\beta$, is the non-perturbed version of the continued fraction under study. That the sequential closure of ω_n lies on a line follows from Theorem 8 below. Combining the continued fraction for ω_n with (3.16) and Theorem 7 yields the intriguing equation:

$$\begin{aligned}
&\frac{-\alpha\beta+q_1}{\alpha+\beta+p_1}+\frac{-\alpha\beta+q_2}{\alpha+\beta+p_2}+\cdots+\frac{-\alpha\beta+q_k}{\alpha+\beta+p_k}\\
&=\lim_{n\to\infty}\frac{-\alpha\beta+q_1}{\alpha+\beta+p_1}+\frac{-\alpha\beta+q_2}{\alpha+\beta+p_2}+\cdots\\
&\qquad\cdots\frac{-\alpha\beta+q_{n-1}}{\alpha+\beta+p_{n-1}}+\frac{-\alpha\beta+q_n}{p_n}-\underbrace{\frac{-\alpha\beta}{\alpha+\beta}+\frac{-\alpha\beta}{\alpha+\beta}+\cdots\frac{-\alpha\beta}{\alpha+\beta}}_{n-k-1\text{ terms}}.
\end{aligned}\tag{3.17}$$

The continued fraction on the left hand side is divergent, while its transformed version on the right hand side converges to the kth approximant of the continued fraction on the left. (3.17) is naturally valid under the condition of Theorem 7 and can be viewed as a continued fraction manifestation of Theorem 4.

(iii) We have assumed that $|\alpha|=|\beta|=1$ and $\alpha\neq\beta$. Actually, by using an equivalence transformation on the continued fraction (3.6), the theorem can be applied under the weaker assumptions $|\alpha|=|\beta|$ and $\alpha\neq\beta$ to yield asymptotics for the approximants in these cases. To be more specific, if $c_i\neq 0$ for $i\geq 1$, then the continued fractions

$$\frac{a_1}{b_1}+\frac{a_2}{b_2}+\frac{a_3}{b_3}+\cdots\tag{3.18}$$

and

$$\frac{c_1a_1}{c_1b_1}+\frac{c_1c_2a_2}{c_2b_2}+\frac{c_2c_3a_3}{c_3b_3}+\cdots\tag{3.19}$$

are said to be *equivalent* since they have the same set of approximants.

The continued fraction at (3.18) is called a *limit 1-periodic continued fraction* when $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, for some $a, b \in \mathbb{C}$.

The associated linear fractional transformation for the continued fraction above is

$$\frac{a}{z+b}.$$

Denote the fixed points of this transformation by

$$z_1 := \frac{b+\sqrt{b^2+4a}}{2}, \qquad z_2 := \frac{b-\sqrt{b^2+4a}}{2}.$$

The continued fraction at (3.18) is said to be a limit 1-periodic continued fraction of *elliptic type* when $z_1 \neq z_2$, but $|z_1| = |z_2|$, see [19].

We consider the case where the continued fraction at (3.18) is a limit 1-periodic continued fraction of elliptic type and, in addition,

$$\sum_{n\geq 1} |a_n - a| < \infty, \qquad \sum_{n\geq 1} |b_n - b| < \infty.$$

Set

$$d := \left|\frac{b+\sqrt{b^2+4a}}{2}\right| = \left|\frac{b-\sqrt{b^2+4a}}{2}\right|,$$

and define

$$\alpha = \frac{b+\sqrt{b^2+4a}}{2d}, \qquad \beta = \frac{b-\sqrt{b^2+4a}}{2d}.$$

Then $\alpha \neq \beta$, $|\alpha| = |\beta| = 1$. Define, for $n \geq 1$, p_n and q_n by

$$a_n = a + p_n, \qquad b_n = b + q_n.$$

Thus

$$K_{n=1}^{\infty} \frac{a+q_n}{b+p_n} = d\, K_{n=1}^{\infty} \frac{-\alpha\beta + q_n/d^2}{\alpha+\beta+p_n/d},$$

this equality following upon setting $c_i = 1/d$ in (3.19). The second continued fraction satisfies the conditions of Theorem 7. Thus this theorem can be applied to all limit 1-periodic continued fractions of elliptic type with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, providing $\sum_{n\geq 1} |a_n - a| < \infty$ and $\sum_{n\geq 1} |b_n - b| < \infty$. Our theorem thus gives detailed information about limit 1-periodic continued fractions of elliptic type of this general class.

Of course, it is known that without any restrictions on how the limit periodic sequences tend to their limits, the behavior can be quit complicated, see [19].

(iv) Our last remark showed that we can loosen the assumption that $|\alpha| = |\beta| = 1$ and $\alpha \neq \beta$ to just $|\alpha| = |\beta|$ and $\alpha \neq \beta$. In fact, suitably interpreted, the asymptotic formula for the f_n in the theorem *continues* to hold even when $|\alpha| \neq |\beta|$. This is the loxodromic case.

To see this, recall the asymptotic for f_n in the theorem in the following form:

$$f_n \sim \frac{a(\alpha/\beta)^{n+1}+b}{c(\alpha/\beta)^{n+1}+d}. \tag{3.20}$$

Clearly when $|\alpha|<|\beta|$ and $n\to\infty$ the right hand side tends to $b/d=h(0)$, assuming that this makes sense. Thus the conclusion of the theorem would be:

$$f_n \sim \frac{b}{d}. \tag{3.21}$$

However, in this case, the continued fraction is limit 1-periodic of loxodromic type, and thus converges to a value f. Moreover, it has attractive fixed point $-\alpha$. In this situation it is well-known, see [19], that its modified limit $f_n(-\alpha)$ also tends to f. But $f_n(-\alpha)$ is exactly the modified continued fraction (3.15) converging to $h(0)=b/d$. Thus (3.21) holds if b/d is defined by (3.15). Obviously the same argument holds when $|\alpha|>|\beta|$, only with the limit $a/c=h(\infty)$ in this case. Thus, interpreting the right hand side of (3.20) as the convergent limit (3.15) for b/d when $|\alpha|<|\beta|$ and as the convergent limit (3.14) for a/c when $|\alpha|>|\beta|$, it follows that (3.20) is true for all finite $\alpha\neq\beta$.

If $\beta=\alpha$, then the continued fraction at (3.6) is equivalent to one of the form $K_{n=1}^{\infty}a_n/1$, where $a_n\to -1/4$. The convergence of continued fractions of this type were studied in [14], [15], [16] and [20]. We remark in passing that they may converge or diverge, depending on the direction and speed of convergence of the a_n to $-1/4$ (see [19], page 158).

Proof of Theorem 7. Define

$$D_n := \begin{pmatrix}\alpha+\beta+p_n & 1\\ -\alpha\beta+q_n & 0\end{pmatrix}, \qquad M := \begin{pmatrix}\alpha+\beta & 1\\ -\alpha\beta & 0\end{pmatrix}. \tag{3.22}$$

For later use, note that

$$M=\begin{pmatrix}-\beta^{-1} & -\alpha^{-1}\\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}-\beta^{-1} & -\alpha^{-1}\\ 1 & 1\end{pmatrix}^{-1}, \tag{3.23}$$

that for $n\in\mathbb{Z}$

$$M^n=\begin{pmatrix}\alpha^{n+1}-\beta^{n+1} & \alpha^n-\beta^n\\ & \\ -\alpha\beta\left(\alpha^n-\beta^n\right) & \alpha\beta^n-\alpha^n\beta\end{pmatrix}\frac{1}{\alpha-\beta}, \tag{3.24}$$

and that for $n\in\mathbb{Z}$

$$M^{-n}=\begin{pmatrix}\alpha^{n-1}-\beta^{n-1} & \dfrac{\alpha^n-\beta^n}{\alpha\beta}\\ & \\ \beta^n-\alpha^n & \dfrac{\beta^{n+1}-\alpha^{n+1}}{\alpha\beta}\end{pmatrix}g_n, \tag{3.25}$$

where, to save space later, we have put $g_n=(\alpha^{1-n}\beta^{1-n})/(\beta-\alpha)$.

Clearly

$$||D_n - M||_\infty = \max\{|p_n|, |q_n|\}.$$

and thus

$$\sum_{n\geq 1} ||D_n - M||_\infty < \infty.$$

It follows that the matrix M and the matrices D_n satisfy the conditions of Theorem 4.

Let P_n and Q_n denote the nth numerator and denominator convergents of the continued fraction (3.6). By the correspondence between matrices and continued fractions (3.1),

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \prod_{j=1}^{n} D_j, \tag{3.26}$$

and using Theorem 4, there exists $F \in GL_2(\mathbb{C})$ defined by

(3.27)

$$F = \lim_{n\to\infty} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \prod_{j=1}^{n} D_j M^{-n}$$

$$= \lim_{n\to\infty} \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} M^{-n}$$

(3.28)

$$= \lim_{n\to\infty} \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} \begin{pmatrix} -\beta^{-1} & -\alpha^{-1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-n} & 0 \\ 0 & \beta^{-n} \end{pmatrix} \begin{pmatrix} -\beta^{-1} & -\alpha^{-1} \\ 1 & 1 \end{pmatrix}^{-1}$$

(3.29)

$$= \lim_{n\to\infty} \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} \begin{pmatrix} \alpha^{n-1} - \beta^{n-1} & \dfrac{\alpha^n - \beta^n}{\alpha\beta} \\ \beta^n - \alpha^n & \dfrac{\beta^{n+1} - \alpha^{n+1}}{\alpha\beta} \end{pmatrix} \frac{\alpha^{1-n}\beta^{1-n}}{\beta - \alpha}.$$

Taking determinants in (3.28) gives an expression for $det(F)$:

$$F_{1,1}F_{2,2} - F_{1,2}F_{2,1} = -\lim_{n\to\infty} (P_nQ_{n-1} - P_{n-1}Q_n)\frac{1}{(\alpha\beta)^n} = -\prod_{n=1}^{\infty}\left(1 - \frac{q_n}{\alpha\beta}\right).$$

The last equality follows from the determinant formula for continued fractions (3.3). Note that $q_n \neq \alpha\beta$ implies that $P_nQ_{n-1} - P_{n-1}Q_n \neq 0$, for $n \geq 1$.

Let $f : GL_2(\mathbb{C}) \to \widehat{\mathbb{C}}$ be given by

$$f : \begin{pmatrix} u & v \\ w & x \end{pmatrix} \mapsto \frac{u}{w}.$$

Note that f is continuous, and thus using Theorem 4, is uniformly continuous on the compact set

$$\overline{\{FM^n : n \geq 1\}} \cup \bigcup_{n\geq 1} \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix}.$$

Theorem 4 and the matrix product representation of continued fractions then give that

$$\frac{P_n}{Q_n} \sim f\left(FM^n\right).$$

Hence using (3.24) and the definition of f,

$$\begin{aligned} \frac{P_n}{Q_n} &\sim \frac{F_{1,1}(\alpha^{n+1} - \beta^{n+1}) + F_{1,2}(-\alpha\beta\,(\alpha^n - \beta^n))}{F_{2,1}(\alpha^{n+1} - \beta^{n+1}) + F_{2,2}(-\alpha\beta\,(\alpha^n - \beta^n))} \\ &= \frac{(F_{1,1} - \beta F_{1,2})\left(\frac{\alpha}{\beta}\right)^{n+1} + (\alpha F_{1,2} - F_{1,1})}{(F_{2,1} - \beta F_{2,2})\left(\frac{\alpha}{\beta}\right)^{n+1} + (\alpha F_{2,2} - F_{2,1})} \\ &= h(\lambda^{n+1}), \end{aligned} \tag{3.30}$$

where

$$h(z) = \frac{az+b}{cz+d}, \tag{3.31}$$

with $a = F_{1,1} - \beta F_{1,2}$, $b = \alpha F_{1,2} - F_{1,1}$, $c = F_{2,1} - \beta F_{2,2}$, $d = \alpha F_{2,2} - F_{2,1}$, and $F_{i,j} \in \mathbb{C}$ are the elements of F. The limit expressions for a, b, c, and d in the theorem follow by simplifying the constants in h defined here, and then using (3.29). The non-vanishing and the product formula for $ad - bc$ follow immediately from the product for $\det(F)$ above and the expressions for a, b, c, and d. Note that we can compactly express the definition of a, b, c, and d in the following matrix equation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} F_{1,1} & F_{1,2} \\ F_{2,1} & F_{2,2} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -\beta & \alpha \end{pmatrix}.$$

Solving for F gives

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ \beta & 1 \end{pmatrix} \frac{1}{\alpha - \beta}. \tag{3.32}$$

Now $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a continuous bijection (since $\det(h) \neq 0$), and thus by the remarks in the introduction,

$$s.c.\left(\frac{P_n}{Q_n}\right) = s.c.(h(\lambda^{n+1})) = h(s.c.(\lambda^{n+1})) = h(\mathbb{T}_m).$$

From Theorem 4 (i),

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} \sim FM^n. \tag{3.33}$$

Substituting (3.24) and (3.32) into (3.33) yields

$$\begin{aligned}
&\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} \\
&\quad \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \alpha^{n+1} - \beta^{n+1} & \alpha^n - \beta^n \\ -\alpha\beta(\alpha^n - \beta^n) & \alpha\beta^n - \beta\alpha^n \end{pmatrix} \frac{1}{(\alpha-\beta)^2} \\
&\quad = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha^{n+1} & * \\ \beta^{n+1} & * \end{pmatrix} \frac{1}{\alpha - \beta} \\
&\quad = \begin{pmatrix} a\alpha^{n+1} + b\beta^{n+1} & * \\ c\alpha^{n+1} + d\beta^{n+1} & * \end{pmatrix} \frac{1}{\alpha - \beta}.
\end{aligned}$$

Thus the sequences P_n and Q_n have the claimed asymptotics by Theorem 4.

Finally, put $A_n = a\alpha^n + b\beta^n$, $B_n = c\alpha^n + d\beta^n$, and observe that

$$\begin{aligned}
\left|f_n - h(\lambda^{n+1})\right| = \left|\frac{P_n}{Q_n} - \frac{A_n}{B_n}\right| &\le \left|\frac{P_nB_n - A_nB_n}{Q_nB_n}\right| + \left|\frac{A_nB_n - Q_nA_n}{Q_nB_n}\right| \\
&\le \left|\frac{1}{Q_n}\right| \varepsilon_n + \left|\frac{A_n}{Q_nB_n}\right| \varepsilon_n,
\end{aligned}$$

and this error is $O(\varepsilon_n)$ providing that B_n is bounded away from 0. (Recall that $Q_n \sim B_n/(\alpha - \beta)$.) It is easy to see that B_n is bounded away from 0 under precisely the two conditions given in the theorem. □

Proof of Corollary 3. (3.14) and (3.15) follow immediately from the value of a modified continued fraction (3.5), with $\omega_n = -\beta$ and $\omega_n = -\alpha$, respectively, and the limit expressions for a, b, c, and d.

To get (3.16), observe that

$$\begin{aligned}
&h(\lambda^{k+1}) = f(FM^k) = f\left(\lim_{n\to\infty} \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} M^{-n} M^k\right) \\
&= f\left(\lim_{n\to\infty} \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} M^{-(n-k)}\right) \\
&= f\left(\lim_{n\to\infty} \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} \begin{pmatrix} \alpha^{n-k-1} - \beta^{n-k-1} & \dfrac{\alpha^{n-k} - \beta^{n-k}}{\alpha\beta} \\ \beta^{n-k} - \alpha^{n-k} & \dfrac{\beta^{n-k+1} - \alpha^{n-k+1}}{\alpha\beta} \end{pmatrix} g_{n-k}\right) \\
&= \lim_{n\to\infty} \frac{(\alpha^{n-k-1} - \beta^{n-k-1})P_n - (\alpha^{n-k} - \beta^{n-k})P_{n-1}}{(\alpha^{n-k-1} - \beta^{n-k-1})Q_n - (\alpha^{n-k} - \beta^{n-k})Q_{n-1}}
\end{aligned}$$

$$= \lim_{n\to\infty} \frac{P_n - \dfrac{\alpha^{n-k}-\beta^{n-k}}{\alpha^{n-k-1}-\beta^{n-k-1}}P_{n-1}}{Q_n - \dfrac{\alpha^{n-k}-\beta^{n-k}}{\alpha^{n-k-1}-\beta^{n-k-1}}Q_{n-1}}$$

$$= \lim_{n\to\infty} \frac{P_n + \omega_{n-k}P_{n-1}}{Q_n + \omega_{n-k}Q_{n-1}},$$

where

$$\omega_j := -\frac{\alpha^j - \beta^j}{\alpha^{j-1}-\beta^{j-1}}.$$

The result now follows from (3.5). □

Proof of Corollary 4. The expression for $h(z)$ follows immediately using algebra from (3.14), (3.15), and (3.16) with $k = -1$. The expressions for a, b, c, and d follow by using (3.12) along with the fact that the coefficients in the two expressions for the linear fractional transformation must be equal up to a constant factor. □

Note that putting $k = 0$ and $k = -1$ in (3.30) gives the following identities:

$$h(\lambda) = \frac{F_{1,1}}{F_{2,1}}, \tag{3.34}$$

$$h(1) = \frac{F_{1,2}}{F_{2,2}}.$$

Let $\mathbb{T}'$ denote the image of $\mathbb{T}$ under h, that is, the sequential closure of the sequence $\{f_n\}$. The asymptotic for f_n given in Theorem 7 is

$$f_n \sim h(\lambda^{n+1}), \tag{3.35}$$

where h is the linear fractional transformation defined in the theorem.

Some observations can immediately be made. It is well known that when λ is not a root of unity, λ^{n+1} is uniformly distributed on $\mathbb{T}$. However, the linear fractional transformation h stretches and compresses arcs of the circle $\mathbb{T}$, so that the distribution of $h(\lambda^{n+1})$ in arcs of $\mathbb{T}'$ is no longer uniform. (Recall uniform distribution on a curve happens when as $n \to \infty$ each segment of the curve get's the proportion of the first n points equal to the ratio of the segment's length to the length of the whole curve.) Additionally, $\mathbb{T}'$ may not be compact in $\mathbb{C}$. So we consider a probability measure on $\mathbb{T}'$ giving the probability of an element $h(\lambda^{n+1})$ being contained in a subset of $\mathbb{T}'$. This measure is easy to write down. Let $S \subset \mathbb{T}'$, then $h^{-1}(S)$ is a subset of the unit circle. Then since λ^n is uniformly distributed on $\mathbb{T}$, $P(S) := \mu(h^{-1}(S))/2\pi$ gives the probability that for any n, $h(\lambda^n) \in S$. Here μ denotes the Lebesgue measure on $\mathbb{T}$. Note that P depends entirely on h, and thus only on the parameters a, b, c, and d.

In general $f_n \notin \mathbb{T}'$, but because of (3.35), as $n \to \infty$, the terms of the sequence f_n get closer and closer to the sequence $h(\lambda^{n+1})$ which lies on $\mathbb{T}'$.

Thus we speak of P as the *limiting probability measure for the sequence* f_n *with respect to* $\mathbb{T}'$.

More specifically, (3.35) implies that there is a one-to-one correspondence between the convergent subsequences of $h(\lambda^{n+1})$ and those of f_n such that the corresponding subsequences tend to the same limit. As h is a homeomorphism and λ^n is uniformly distributed on $\mathbb{T}$, it follows that the probability of an element of $s.c.(f_n)$ being contained in a subset S of $\mathbb{T}'$ is exactly $P(S) = \mu(h^{-1}(S))/2\pi$.

Fortunately, this distribution is completely controlled by the known parameters a, b, c, and d. The following theorem gives the points on the sequential closures whose neighborhood arcs have the greatest and least concentrations of approximants.

Theorem 8. *When $m = \infty$ and $cd \neq 0$, the points on*

$$\frac{a\mathbb{T} + b}{c\mathbb{T} + d}$$

with the highest and lowest concentrations of approximants are

$$\frac{\frac{a}{c}|c| + \frac{b}{d}|d|}{|c| + |d|} \qquad \text{and} \qquad \frac{-\frac{a}{c}|c| + \frac{b}{d}|d|}{-|c| + |d|},$$

respectively. If either $c = 0$ or $d = 0$, then all points on the sequential closure have the same concentration. The radius of the sequential closure circle in $\mathbb{C}$ is

$$\left| \frac{\alpha - \beta}{|c|^2 - |d|^2} \prod_{n=1}^{\infty} \left(1 - \frac{q_n}{\alpha\beta} \right) \right|,$$

and its center is the complex point

$$\frac{|h(1)|^2(h(-1) - h(i)) + |h(-1)|^2(h(i) - h(1)) + |h(i)|^2(h(1) - h(-1))}{h(1)(\overline{h(i)} - \overline{h(-1)}) + h(-1)(\overline{h(1)} - \overline{h(i)}) + h(i)(\overline{h(-1)} - \overline{h(1)})}.$$

The sequential closure is a line in $\mathbb{C}$ if and only if $|c| = |d|$, and in this case the point of least concentration is ∞.

Proof. Let $g(\theta) = h(e^{i\theta})$. Thus $g(\theta)$ parametrizes $\mathbb{T}'$ for $\theta \in [0, 2\pi]$ and $e^{i\theta}$ moves with a uniform speed around $\mathbb{T}$ as θ moves uniformly from 0 to 2π. Then $g(\theta)$ moves around $\mathbb{T}'$ at different speeds depending on how the length $g(\theta)$ change with θ. Accordingly, we wish to compute the rate of change of the length of $g(\theta)$ with respect to θ. We then wish to know when this value is minimum and maximum. To this end put

$$l(\theta) := \int_0^\theta |g'(\theta)| d\theta.$$

Accordingly, $l'(\theta) = |g'(\theta)|$. An easy computation gives

$$l'(\theta) = \frac{|ad - bc|}{|c|^2 + |d|^2 + c\overline{d}e^{i\theta} + \overline{c}de^{-i\theta}},$$

and thus

$$l''(\theta) = i\frac{|ad-bc|(\bar{c}de^{-i\theta} - c\bar{d}e^{i\theta})}{(|c|^2+|d|^2+c\bar{d}e^{i\theta}+\bar{c}de^{-i\theta})^2}.$$

Clearly $l''(\theta) = 0$ if and only if $e^{i\theta} = \pm|c|d/c|d|$. Plugging these values into h gives the points where the length of $g(\theta)$ is changing most and least with respect to θ.

To compute the radius of $\mathbb{T}'$, one computes $l(2\pi)/2\pi$:

$$\begin{aligned}\frac{l(2\pi)}{2\pi} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{|ad-bc|}{|c|^2+|d|^2+c\bar{d}e^{i\theta}+\bar{c}de^{-i\theta}}d\theta \\ &= \frac{|ad-bc|}{2\pi i}\oint \frac{dz}{(c+dz)(\bar{d}+\bar{c}z)},\end{aligned}$$

where the contour on the last integral is the unit circle. A routine evaluation by the residue theorem along with (3.12) gives the result. The center can easily be computed as it is the circumcenter of the triangle formed by any three points on the circle, for example, $z_1 = h(1)$, $z_2 = h(-1)$, and $z_3 = h(i)$. The well-known formula for the circumcenter of three non-collinear points in the complex plane

$$\frac{|z_1|^2(z_2-z_3)+|z_2|^2(z_3-z_1)+|z_3|^2(z_1-z_2)}{z_1(\overline{z_3}-\overline{z_2})+z_2(\overline{z_1}-\overline{z_3})+z_3(\overline{z_2}-\overline{z_1})}$$

thus gives the center of the sequential closure circle. The final conclusions of the theorem follow immediately from the formulas for the points of highest and lowest concentration. □

Corollary 5. *If the sequential closure of the continued fraction in* (3.6) *is a line in* $\mathbb{C}$*, then the point of highest concentration of approximants in the sequential closure is exactly*

$$\frac{h(\infty)+h(0)}{2} = \frac{1}{2}\left(\frac{a}{c}+\frac{b}{d}\right),$$

the average of the first two modifications of (3.6) *given in Corollary 3. Moreover, if the sequential closure is* $\mathbb{R}$*, then the limiting probability density function for the approximants is given by*

$$p(x) = \frac{h(\infty)-h(0)}{2\pi i(x-h(\infty))(x-h(0))}. \tag{3.36}$$

Proof. If the sequential closure is a line, then Theorem 8 implies that $|c| = |d|$. The same theorem also implies that the point of highest concentration is given by

$$\frac{\frac{a}{c}|c|+\frac{b}{d}|d|}{|c|+|d|}.$$

When $|c| = |d|$, this simplifies to

$$\frac{1}{2}\left(\frac{a}{c}+\frac{b}{d}\right),$$

which is the average of $h(\infty)$ and $h(0)$.

Suppose the sequential closure is $\mathbb{R}$. Let the point $x \in \mathbb{R}$ be related to the point z on the unit circle via

$$x = h(z) = \frac{az+b}{cz+d},$$

and suppose $z = e^{i\theta}$. Let $\theta_0 \in (0, 2\pi]$ be the angle for which z is mapped to ∞ by $h(z)$, and put $z_0 = e^{i\theta_0}$. Let $p(x)$ denote the probability density function and let f_i denote the i-th approximant of (3.6). Then for any interval $[a, b]$,

$$\begin{aligned}\int_a^b p(x)dx &= \lim_{n\to\infty} \frac{\#\{f_i \in [a,b]\}_{0\le i\le n}}{n} \\ &= \frac{\mu(h^{-1}([a,b]))}{2\pi},\end{aligned}$$

where, the second equality follows from remarks made in the discussion preceding Theorem 8. In particular,

$$\int_{-\infty}^x p(t)dt = \frac{\text{length of the arc clockwise from } z_0 \text{ to } z}{2\pi} = \frac{\theta_0 - \theta}{2\pi}.$$

Using the Fundamental Theorem of Calculus, one obtains

$$p(x) = \frac{-1}{2\pi}\frac{d\theta}{dx} = \frac{-1}{2\pi i z}\frac{dz}{dx} = \frac{ad-bc}{2\pi i(cx-a)(dx-b)}.$$

The result now follows from the definition of $h(z)$. □

It is also possible to derive convergent continued fractions which have the same limit as the modified continued fractions in Theorem 7.

Corollary 6. *Let α, β, $\{p_n\}$, $\{q_n\}$, $h(z)$, and the matrix F be as in Theorem 7 and its proof. Then*

$$h(\infty) = -\beta + \frac{q_1+\beta p_1}{\alpha+p_1} + \frac{(q_1-\alpha\beta)(q_2+\beta p_2)}{(\alpha+p_2)(q_1+\beta p_1)+\beta(q_2+\beta p_2)} + K_{n=3}^{\infty}\frac{(q_{n-1}-\alpha\beta)(q_n+\beta p_n)(q_{n-2}+\beta p_{n-2})}{(\alpha+p_n)(q_{n-1}+\beta p_{n-1})+\beta(q_n+\beta p_n)}, \tag{3.37}$$

$$h(0) = -\alpha + \frac{q_1+\alpha p_1}{\beta+p_1} + \frac{(q_1-\alpha\beta)(q_2+\alpha p_2)}{(\beta+p_2)(q_1+\alpha p_1)+\alpha(q_2+\alpha p_2)} + K_{n=3}^{\infty}\frac{(q_{n-1}-\alpha\beta)(q_n+\alpha p_n)(q_{n-2}+\alpha p_{n-2})}{(\beta+p_n)(q_{n-1}+\alpha p_{n-1})+\alpha(q_n+\alpha p_n)}. \tag{3.38}$$

Let $k \in \mathbb{Z}$ and assume that α/β is not a root of unity. Set

$$\omega_n = -\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha^{n-k-1}-\beta^{n-k-1}}, \qquad \text{for } n \ge k' := \max\{3, k+3\}.$$

Then

$$h(\lambda^{k+1}) = \frac{-\alpha\beta + q_1}{\alpha+\beta+p_1} + \cdots + \frac{-\alpha\beta + q_{k'-1}}{\alpha+\beta+p_{k'-1}} + \frac{-\alpha\beta + q_{k'}}{\alpha+\beta+p_{k'}+\omega_{k'}} + \frac{-\alpha\beta + q_{k'+1} - \omega_{k'}(\alpha+\beta+p_{k'+1}+\omega_{k'+1})}{\alpha+\beta+p_{k'+1}+\omega_{k'+1}} + K_{n=k'+2}^{\infty}\frac{c_n}{d_n}, \tag{3.39}$$

where

$$c_n = (q_{n-1}-\alpha\beta)\,\frac{-\alpha\beta+q_n-\omega_{n-1}\,(\alpha+\beta+p_n+\omega_n)}{-\alpha\beta+q_{n-1}-\omega_{n-2}\,(\alpha+\beta+p_{n-1}+\omega_{n-1})}$$

$$d_n = \alpha+\beta+p_n+\omega_n-\omega_{n-2}\frac{-\alpha\beta+q_n-\omega_{n-1}\,(\alpha+\beta+p_n+\omega_n)}{-\alpha\beta+q_{n-1}-\omega_{n-2}\,(\alpha+\beta+p_{n-1}+\omega_{n-1})}.$$

Proof. The continued fraction (3.37) above is equivalent (after a sequence of similarity transformations have been applied to simplify it) to the Bauer-Muir transformation (see [19], page 76, for example) of the continued fraction

$$\frac{-\alpha\beta + q_1}{\alpha+\beta+p_1} + \frac{-\alpha\beta + q_2}{\alpha+\beta+p_2} + \frac{-\alpha\beta + q_3}{\alpha+\beta+p_3} + \frac{-\alpha\beta + q_4}{\alpha+\beta+p_4} + \cdots \tag{3.40}$$

with respect to the sequence $\omega_n = -\beta$, $n \geq 0$. This in turn equals $h(\infty)$ by (3.14).

The continued fraction at (3.38) is likewise equivalent to the Bauer-Muir transformation of (3.40) with respect to the sequence $\omega_n = -\alpha$, $n \geq 0$. This in turn equals $h(0)$ by (3.15).

The continued fraction at (3.39) above is the Bauer-Muir transformation of the continued fraction

$$\frac{-\alpha\beta + q_1}{\alpha+\beta+p_1} + \frac{-\alpha\beta + q_2}{\alpha+\beta+p_2} + \frac{-\alpha\beta + q_3}{\alpha+\beta+p_3} + \frac{-\alpha\beta + q_4}{\alpha+\beta+p_4} + \cdots$$

with respect to the sequence $\{\omega_n\}$, where $\omega_n = 0$ for $0 \leq n \leq k'-1$ and

$$\omega_n = -\frac{\alpha^n - \beta^n}{\alpha^{n-1}-\beta^{n-1}}, \qquad \text{for } n \geq k'.$$

This in turn equals $h(\lambda^{k+1})$ by (3.16). □

An interesting special case of Theorem 7 occurs when α and β are distinct m-th roots of unity ($m \geq 2$). In this situation the continued fraction

$$\frac{-\alpha\beta + q_1}{\alpha+\beta+p_1} + \frac{-\alpha\beta + q_2}{\alpha+\beta+p_2} + \frac{-\alpha\beta + q_3}{\alpha+\beta+p_3} + \frac{-\alpha\beta + q_4}{\alpha+\beta+p_4} + \cdots$$

becomes limit periodic and the sequences of approximants in the m different arithmetic progressions modulo m converge. The corollary below, which is also proved in [7], is an easy consequence of Theorem 7.

Corollary 7. *Let* $\{p_n\}_{n\geq 1}$, $\{q_n\}_{n\geq 1}$ *be complex sequences satisfying*

$$\sum_{n=1}^{\infty}|p_n| < \infty, \qquad \sum_{n=1}^{\infty}|q_n| < \infty.$$

Let α and β be distinct roots of unity and let m be the least positive integer such that $\alpha^m = \beta^m = 1$. Define

$$G := \frac{-\alpha\beta + q_1}{\alpha + \beta + p_1} + \frac{-\alpha\beta + q_2}{\alpha + \beta + p_2} + \frac{-\alpha\beta + q_3}{\alpha + \beta + p_3} + \cdots.$$

Let $\{P_n/Q_n\}_{n=1}^{\infty}$ denote the sequence of approximants of G. If $q_n \neq \alpha\beta$ for any $n \geq 1$, then G does not converge. However, the sequences of numerators and denominators in each of the m arithmetic progressions modulo m do converge. More precisely, there exist complex numbers $A_0, \ldots, A_{m-1}$ and $B_0, \ldots, B_{m-1}$ such that, for $0 \leq i < m$,

$$\lim_{k\to\infty} P_{m\,k+i} = A_i, \qquad \lim_{k\to\infty} Q_{m\,k+i} = B_i. \tag{3.41}$$

Extend the sequences $\{A_i\}$ and $\{B_i\}$ over all integers by making them periodic modulo m so that (3.41) continues to hold. Then for integers i,

$$A_i = \left(\frac{A_1 - \beta A_0}{\alpha - \beta}\right)\alpha^i + \left(\frac{\alpha A_0 - A_1}{\alpha - \beta}\right)\beta^i, \tag{3.42}$$

and

$$B_i = \left(\frac{B_1 - \beta B_0}{\alpha - \beta}\right)\alpha^i + \left(\frac{\alpha B_0 - B_1}{\alpha - \beta}\right)\beta^i. \tag{3.43}$$

Moreover,

$$A_i B_j - A_j B_i = -(\alpha\beta)^{j+1}\frac{\alpha^{i-j} - \beta^{i-j}}{\alpha - \beta}\prod_{n=1}^{\infty}\left(1 - \frac{q_n}{\alpha\beta}\right). \tag{3.44}$$

Put $\alpha := \exp(2\pi i a/m)$, $\beta := \exp(2\pi i b/m)$, $0 \leq a < b < m$, and $r := m/\gcd(b-a, m)$. Then G has r distinct limits in $\widehat{\mathbb{C}}$ which are given by A_j/B_j, $1 \leq j \leq r$. Finally, for $k \geq 0$ and $1 \leq j \leq r$,

$$\frac{A_{j+kr}}{B_{j+kr}} = \frac{A_j}{B_j}.$$

Remark: We refer to the number r in the corollary as the *rank* of the continued fraction.

Proof. Let M be as in Theorem 7. It follows from (3.23) that

$$M^j = \begin{pmatrix} \dfrac{\alpha^{1+j} - \beta^{1+j}}{\alpha - \beta} & \dfrac{\alpha^j - \beta^j}{\alpha - \beta} \\ -\dfrac{\alpha\,\beta\,\left(\alpha^j - \beta^j\right)}{\alpha - \beta} & \dfrac{-\alpha^j\,\beta + \alpha\,\beta^j}{\alpha - \beta} \end{pmatrix}, \tag{3.45}$$

and thus that

$$M^m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M^j \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad 1 \leq j < m.$$

Let the matrix F be as in Theorem 7. From the second equality at (3.29), we have that

$$\lim_{n\to\infty}\begin{pmatrix} P_{mn+i} & P_{mn+i-1} \\ Q_{mn+i} & Q_{mn+i-1}\end{pmatrix} = \lim_{n\to\infty} F\,M^{mn+i} = F\,M^{i}. \tag{3.46}$$

This proves (3.41).

Now let $A_i := \lim_{n\to\infty} P_{mn+i}$, and $B_i := \lim_{n\to\infty} Q_{mn+i}$. Notice by definition that the sequences $\{A_i\}$ and $\{B_i\}$ are periodic modulo m. It easily follows from (3.46) that

$$\begin{pmatrix} A_i & A_{i-1} \\ B_i & B_{i-1}\end{pmatrix} = \begin{pmatrix} A_j & A_{j-1} \\ B_j & B_{j-1}\end{pmatrix} M^{i-j}.$$

(3.45) also gives that

$$A_i = A_j\frac{\alpha^{1+i-j}-\beta^{1+i-j}}{\alpha-\beta} - A_{j-1}\frac{\alpha\,\beta\,\left(\alpha^{i-j}-\beta^{i-j}\right)}{\alpha-\beta}, \tag{3.47}$$

and

$$B_i = B_j\frac{\alpha^{1+i-j}-\beta^{1+i-j}}{\alpha-\beta} - B_{j-1}\frac{\alpha\,\beta\,\left(\alpha^{i-j}-\beta^{i-j}\right)}{\alpha-\beta}. \tag{3.48}$$

Thus

$$A_iB_j - A_jB_i = \frac{\left(A_j\,B_{-1+j} - A_{-1+j}\,B_j\right)\,\alpha\,\beta\,\left(\alpha^{i-j}-\beta^{i-j}\right)}{\alpha-\beta}.$$

Equations (3.42) and (3.43) follow from (3.47) and (3.48) by setting $j = 1$. (3.44) follows after applying the determinant formula

$$\begin{aligned} A_jB_{j-1} - A_{j-1}B_j &= -\lim_{k\to\infty}\prod_{n=1}^{mk+j}(\alpha\beta - q_n) \\ &= -(\alpha\beta)^j\prod_{n=1}^{\infty}\left(1-\frac{q_n}{\alpha\beta}\right). \end{aligned}$$

Since $\sum_{j=1}^{\infty}|q_j|$ converges to a finite value, the infinite product on the right side converges.

For the continued fraction to converge, $A_iB_{i-1} - A_{i-1}B_i = 0$ is required. However, (3.44) shows that this is not the case. □

3.1. **Computing subsequences of approximants converging to any point on the sequential closure.** We recall one of the main conclusions of Theorem 7. Namely, that if $\sum|p_n| < \infty$, $\sum|q_n| < \infty$, $|\alpha| = |\beta| = 1$ and $\lambda := \alpha/\beta$ is not a root of unity, then the n-th approximant of $K(-\alpha\beta + q_n)/(\alpha\beta + p_n)$, f_n, satisfies

$$f_n \sim h(\lambda^{n+1}) := \frac{a\lambda^{n+1}+b}{c\lambda^{n+1}+d},$$

for some a, b, c and $d \in \mathbb{C}$. Thus the approximants densely approach a circle in the complex plane and a natural question is the following: is it possible explicitly to determine a subsequence of approximants converging to $h(e^{2\pi i\theta})$, for any $\theta \in [0,1)$? Using the regular continued fraction for θ this question is answered in the affirmative with the following algorithm.

Let $\lambda = e^{2\pi i\gamma}$, $\gamma \in (0,1)$ and let $\{a_n/b_n\}$ denote the sequence of even indexed approximants in the regular continued fraction expansion of γ. Since λ is not a root of unity, it follows that γ is irrational. For real z, let $\{z\}$ denote the fractional part of z. Thus $\{z\} = z - \lfloor z \rfloor$. Let $\theta \in [0,1)$ and, for $n \geq 1$, let r_n denote the least positive integer satisfying $0 \leq r_n/b_n - \theta < 1/b_n$. For any positive integer x,

$$x\gamma - \theta = x\left(\gamma - \frac{a_n}{b_n}\right) + \frac{xa_n - r_n}{b_n} + \left(\frac{r_n}{b_n} - \theta\right).$$

Since $\gcd(a_n, b_n) = 1$, there exists a non-negative integer $x < b_n$ satisfying $a_n x \equiv r_n \pmod{b_n}$. Let k_n be this solution. Since $(a_n k_n - r_n)/b_n \in \mathbb{Z}$, it follows that

$$\{k_n\lambda - \theta\} = \left\{k_n\left(\gamma - \frac{a_n}{b_n}\right) + \left(\frac{r_n}{b_n} - \theta\right)\right\}.$$

If the sequence $\{k_n\}$ is unbounded, let $\{j_n\}$ be a strictly increasing subsequence. If $\{k_n\}$ is bounded, replace each k_n by $k_n + b_n$ and once again let $\{j_n\}$ be a strictly increasing subsequence. From the theory of regular continued fractions we have that in either case

$$k_n\left|\gamma - \frac{a_n}{b_n}\right| < (k_n + b_n)\left|\gamma - \frac{a_n}{b_n}\right| < \frac{2}{b_n},$$

and thus that

$$\{j_n\gamma - \theta\} \to 0.$$

It now follows that $f_{j_n - 1} \sim h(\gamma^{j_n}) \to h(e^{2\pi i\theta})$. Thus

$$\lim_{n\to\infty} f_{j_n-1} = h(e^{2\pi i\theta}).$$

Note that for rational $\lambda = m/n$, one takes approximants in arithmetic progressions modulo n to obtain the subsequences tending to the discrete sequential closure.

Finally, we briefly compare our results with a theorem of Scott and Wall [26, 35].

Consider the continued fraction

$$\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots. \tag{3.49}$$

Theorem 9 (Scott and Wall). *If the series* $\sum |b_{2p+1}|$ *and* $\sum |b_{2p+1}s_p^2|$, *where* $s_p = b_2 + b_4 + \cdots + b_{2p}$, *converge, and* $\liminf |s_p| < \infty$, *then the continued fraction* (3.49) *diverges. The sequence of its odd numerator and denominators convergents,* $\{A_{2p+1}\}$ *and* $\{B_{2p+1}\}$, *converge to finite limits* F_1 *and* G_1, *respectively. Moreover, if* s *is a finite limit point of the sequence* $\{s_p\}$, *and*

$\lim s_p = s$ as p tends to ∞ over a certain sequence P of indices, then A_{2p} and B_{2p} converge to finite limits $F(s)$ and $G(s)$, respectively as p tends to ∞ over P, and

$$F_1 G(s) - G_1 F(s) = 1.$$

If the sequence $\{s_p\}$ has two different finite limit points s and t, then

$$F(s)G(t) - F(t)G(s) = t - s.$$

Finally, corresponding to values of p for which $\lim s_p = \infty$, we have

$$\lim \frac{A_{2p}}{B_{2p}} = \frac{F_1}{G_1},$$

finite or infinite.

As far as we know, this theorem is closest in theme to the idea of this paper. On the one hand it makes no assumptions about the size of the sequential closure. On the other hand, it retains much of the structure of the Stern-Stolz theorem, in as much as it focuses on the parity of the index of the approximants. To understand sequential closures in general, all subsequences need to be considered. At any rate, Theorem 9 does not focus on the sequential closure, but rather on loosening the l_1 assumption to the subsequence odd indexed elements of the continued fraction.

One naturally wonders just how effectively the parameters a, b, c, and d in Theorem 7 can be computed. In the next section, a particular continued fraction is considered which generalizes one of Ramanujan's, as well as the 3/2 continued fraction given in the introduction, and these parameters explicitly are computed as well-behaved meromorphic functions of the variables in the continued fraction. Thus, for the q-continued fraction studied in the next section, the parameters can not only be computed, but also have nice formulas.

4. A generalization of a Ramanujan Continued Fraction

In this section we study the non-trivial case of Theorem 7 in which the perturbing sequences p_n and q_n are geometric progressions tending to 0. The inspiration for this is the beautiful continued fraction (1.3) of Ramanujan. Our theorem is interesting in that it covers both the loxodromic (convergent) as well as the elliptic (divergent) cases simultaneously. Another point of this section is that it shows how Theorem 7 gives another approach evaluating continued fractions. In fact it is interesting to compare the proof of Theorem 10 to the proofs of special cases given previously by different methods, see [1, 7, 13].

We first recall that a ${}_1\phi_1$ basic hypergeometric series is defined for $|q| < 1$ by

$${}_1\phi_1(a; b; q, x) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n (b;q)_n} (-1)^n q^{n(n-1)/2} x^n.$$

For the q-product notation used here, please see the introduction.

Theorem 10. *Let $|q| < 1$ and $\alpha \neq \beta$ and put $\lambda = \alpha/\beta$. Then,*

$$\frac{-\alpha\beta + xq}{\alpha + \beta + yq} + \frac{-\alpha\beta + xq^2}{\alpha + \beta + yq^2} + \frac{-\alpha\beta + xq^3}{\alpha + \beta + yq^3} + \cdots + \frac{-\alpha\beta + xq^n}{\alpha + \beta + yq^n} \sim \tag{4.1}$$
$$\frac{\left(\frac{xq}{\alpha} - \beta\right) {}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq^2}{\alpha}\right) \lambda^{n+1} - \left(\frac{xq}{\beta} - \alpha\right) {}_1\phi_1\left(\frac{-xq}{y\beta}; \frac{\alpha q}{\beta}; q, \frac{-yq^2}{\beta}\right)}{{}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq}{\alpha}\right) \lambda^{n+1} - {}_1\phi_1\left(\frac{-xq}{y\beta}; \frac{\alpha q}{\beta}; q, \frac{-yq}{\beta}\right)}.$$

Finally, assuming $|\alpha| = |\beta| = 1$, let the order of λ in $\mathbb{T}$ be $m \neq 1$. Then,

$$s.c.\left(\frac{-\alpha\beta + xq}{\alpha + \beta + yq} + \frac{-\alpha\beta + xq^2}{\alpha + \beta + yq^2} + \frac{-\alpha\beta + xq^3}{\alpha + \beta + yq^3} + \cdots\right) \tag{4.2}$$
$$= \frac{\left(\frac{xq}{\alpha} - \beta\right) {}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq^2}{\alpha}\right) \mathbb{T}_m - \left(\frac{xq}{\beta} - \alpha\right) {}_1\phi_1\left(\frac{-xq}{y\beta}; \frac{\alpha q}{\beta}; q, \frac{-yq^2}{\beta}\right)}{{}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq}{\alpha}\right) \mathbb{T}_m - {}_1\phi_1\left(\frac{-xq}{y\beta}; \frac{\alpha q}{\beta}; q, \frac{-yq}{\beta}\right)}.$$

Note that when $|\alpha| \neq |\beta|$, (4.1) shows that the continued fraction converges and provides its limit. It is also in agreement with remark (v) following Theorem 7. For this theorem, we have not provided the error term for the difference between the left and right hand sides of (4.1). But Theorem 7 implies that in the elliptic case (when $|\alpha| = |\beta|$), this error is $O(q^n)$. In the loxodromic case ($|\alpha| \neq |\beta|$), the error term can be computed from Corollary 11 in Chapter IV of [19].

Before preceeding with the proof, we note a couple of corollaries. Theorem 10 generalizes many well-known continued fraction evaluations. For example, setting $\alpha = y = 0$ and $\beta = 1$, dividing by x, changing x to x/q, taking reciprocals, and letting $n \to \infty$ in (4.1) yields the well-known evaluation of the Rogers-Ramanujan continued fraction:

Corollary 8. *For $x, q \in \mathbb{C}$ and $|q| < 1$,*

$$1 + \frac{xq}{1} + \frac{xq^2}{1} + \cdots = \frac{\sum_{m\geq 0} \frac{q^{m^2} x^m}{(q)_m}}{\sum_{m\geq 0} \frac{q^{m^2+m} x^m}{(q)_m}}.$$

The next corollary generalizes Ramanujan's continued fraction (1.3) with three limits given in the introduction.

Corollary 9. *Let ω be a primitive m-th root of unity and let $\bar{\omega} = 1/\omega$. Let $1 \leq i \leq m$. Then*

$$\lim_{k\to\infty} \frac{1}{\omega + \bar{\omega} + q} - \frac{1}{\omega + \bar{\omega} + q^2} - \cdots \frac{1}{\omega + \bar{\omega} + q^{mk+i}} \tag{4.3}$$
$$= \frac{\omega^{1-i} {}_1\phi_1\left(0; q\omega^2; q, -q^2\omega\right) - \omega^{i-1} {}_1\phi_1\left(0; q/\omega^2; q, -q^2/\omega\right)}{\omega^{-i} {}_1\phi_1\left(0; q\omega^2; q, -q\omega\right) - \omega^{i} {}_1\phi_1\left(0; q/\omega^2; q, -q/\omega\right)}.$$

Proof. This is immediate from (4.1), upon setting $x = 0$, $y = 1$, $\alpha = \omega$, $\beta = \omega^{-1}$, $n = mk + i$, then noting that $\omega^{mk} = 1$. □

Remark: This result in its present form first appeared in [13]. A different proof was given in our paper [7].

We now continue with the proof of Theorem 10. Following the proof, other special cases are studied, including the 3/2 continued fraction from the introduction.

Proof. First consider the case $|\alpha| = |\beta|$. It is convenient to work with the related continued fraction

$$\frac{1}{1}\,+\,\frac{-\alpha\beta + xq}{\alpha+\beta+yq}\,+\,\frac{-\alpha\beta + xq^2}{\alpha+\beta+yq^2}\,+\,\frac{-\alpha\beta + xq^3}{\alpha+\beta+yq^3}\,+\cdots. \tag{4.4}$$

Let A_n and B_n denote the n-th numerator convergent and n-th denominator convergent, respectively, of this continued fraction.

Let M be defined by (3.22) and recall from (3.23) that

$$M = \begin{pmatrix} -\beta^{-1} & -\alpha^{-1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} -\beta^{-1} & -\alpha^{-1} \\ 1 & 1 \end{pmatrix}^{-1}. \tag{4.5}$$

Note that by (3.1), (4.4) corresponds to the matrix product

$$U_n := \begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \prod_{j=1}^{n-1} \begin{pmatrix} \alpha+\beta+yq^j & 1 \\ -\alpha\beta + xq^j & 0 \end{pmatrix}.$$

Put $F_n = U_n M^{-n}$. By Theorem 7, there exists a matrix F defined by $F = \lim_{n\to\infty} F_n$. Following the ideas of Theorem 7, define the sequences G_n and H_n by

$$G_n = \frac{U_{n+1} - \beta U_n}{\alpha^n(\alpha-\beta)}, \qquad H_n = \frac{U_{n+1} - \alpha U_n}{\beta^n(\beta-\alpha)}. \tag{4.6}$$

From (3.11) one can see that $\lim_{n\to\infty} G_n$ and $\lim_{n\to\infty} H_n$ exist. It is clear that

$$U_n = G_n\alpha^n + H_n\beta^n. \tag{4.7}$$

We next determine $\lim_{n\to\infty} G_n$. For $n \geq 1$, let

$$D_n = \begin{pmatrix} \alpha+\beta+yq^n & 1 \\ -\alpha\beta + xq^n & 0 \end{pmatrix}.$$

Since $U_{n+1} = U_n D_n$,

$$\begin{aligned} G_{n+1} &= \frac{U_{n+1}(D_{n+1} - \beta I)}{\alpha^{n+1}(\alpha-\beta)} = \frac{U_n D_n (D_{n+1} - \beta I)}{\alpha^{n+1}(\alpha-\beta)} \\ &= \alpha^{-1} G_n (D_n - \beta I)^{-1} D_n (D_{n+1} - \beta I) \\ &= G_n \begin{pmatrix} 1 + \alpha^{-1}\beta q + \alpha^{-1} y q^{n+1} & \alpha^{-1} \\ -\beta\, q + \alpha^{-1} x q^{n+1} & 0 \end{pmatrix}. \end{aligned}$$

Let

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} := G_n.$$

Then

$$a_{n+1} = (1+\alpha^{-1}\beta q+\alpha^{-1}yq^{n+1})a_n + (-\beta\, q+\alpha^{-1}xq^{n+1})b_n \tag{4.8}$$
$$= (1+\alpha^{-1}\beta q+\alpha^{-1}yq^{n+1})a_n + \alpha^{-1}(-\beta\, q+\alpha^{-1}xq^{n+1})a_{n-1}.$$

We use the generating function $F(t) := \sum_{n=1}^{\infty} a_n t^n$ to find $\lim_{n\to\infty} a_n$. Multiply (4.8) by t^{n+1} and sum over $n \geq 1$ to get

$$F(t) - a_1 t = (1+\alpha^{-1}\beta\, q)tF(t) + \alpha^{-1}q\, ytF(tq) - \alpha^{-1}\beta\, q\, t^2(F(t)+a_0)$$
$$+ \alpha^{-2}xt^2q^2(F(tq)+a_0),$$

or

$$F(t) = \frac{t(a_1+(xq/\alpha-\beta)a_0qt/\alpha)}{(1-t)(1-\beta qt/\alpha)} + \frac{qyt/\alpha(1+xtq/y\alpha)}{(1-t)(1-\beta qt/\alpha)}F(tq). \tag{4.9}$$

Upon iteration (note that $F(0)=0$) this yields
(4.10)

$$F(t) = \frac{1}{q}\sum_{n=1}^{\infty} \frac{t^n(y/\alpha)^{n-1}q^{n(n+1)/2}(-xqt/y\alpha)_{n-1}(a_1+a_0(xq/\alpha-\beta)tq^n/\alpha)}{(t)_n(\beta qt/\alpha)_n}.$$

Since $|q|<1$, this series is convergent and satisfies (4.9). Thus

$$\lim_{n\to\infty} a_n = \lim_{t\to 1_-}(1-t)F(t) \tag{4.11}$$
$$= \frac{1}{q}\sum_{n=1}^{\infty} \frac{(y/\alpha)^{n-1}q^{n(n+1)/2}(-xq/y\alpha)_{n-1}(a_1+a_0(xq/\alpha-\beta)q^n/\alpha)}{(q)_{n-1}(\beta q/\alpha)_n}.$$

We next find a_1 and a_0. From (4.4),

$$U_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$U_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$
$$U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha+\beta+yq & 1 \\ -\alpha\beta+xq & 0 \end{pmatrix} = \begin{pmatrix} yq+\alpha+\beta & 1 \\ xq+yq+\alpha+\beta-\alpha\beta & 1 \end{pmatrix}.$$

From (4.6),

$$G_0 = \frac{U_1-\beta U_0}{\alpha-\beta} = \frac{1}{\alpha-\beta}\begin{pmatrix} 1 & -\beta \\ 1-\beta & 1 \end{pmatrix},$$
$$G_1 = \frac{U_2-\beta U_1}{\alpha(\alpha-\beta)} = \frac{1}{\alpha(\alpha-\beta)}\begin{pmatrix} yq+\alpha & 1 \\ xq+yq+\alpha-\alpha\beta & 1-\beta \end{pmatrix}.$$

Thus,

$$a_0 = \frac{1}{\alpha-\beta}, \qquad a_1 = \frac{yq+\alpha}{\alpha(\alpha-\beta)},$$

and

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \frac{1}{q(\alpha-\beta)}\sum_{n=1}^{\infty}\frac{\left(\frac{y}{\alpha}\right)^{n-1} q^{n(n+1)/2}\left(\frac{-xq}{y\alpha}\right)_{n-1}\left(1+\frac{qy}{\alpha}+\left(\frac{xq}{\alpha}-\beta\right)\frac{q^n}{\alpha}\right)}{(q)_{n-1}(\beta q/\alpha)_n}\\
&= \frac{1}{q(\alpha-\beta)}\sum_{n=1}^{\infty}\frac{(y/\alpha)^{n-1}q^{n(n+1)/2}(-xq/y\alpha)_{n-1}}{(q)_{n-1}(\beta q/\alpha)_{n-1}}\\
&\qquad + \frac{1}{(\alpha-\beta)}\sum_{n=1}^{\infty}\frac{(y/\alpha)^{n}q^{n(n+1)/2}(-xq/y\alpha)_{n}}{(q)_{n-1}(\beta q/\alpha)_{n}}\\
&= \frac{1}{(\alpha-\beta)}\sum_{n=0}^{\infty}\frac{(y/\alpha)^{n}q^{n(n+3)/2}(-xq/y\alpha)_{n}}{(q)_{n}(\beta q/\alpha)_{n}}\\
&\qquad + \frac{1}{(\alpha-\beta)}\sum_{n=1}^{\infty}\frac{(y/\alpha)^{n}q^{n(n+1)/2}(-xq/y\alpha)_{n}(1-q^n)}{(q)_{n}(\beta q/\alpha)_{n}}\\
&= \frac{1}{(\alpha-\beta)}\sum_{n=0}^{\infty}\frac{(y/\alpha)^{n}q^{n(n+1)/2}(-xq/y\alpha)_{n}}{(q)_{n}(\beta q/\alpha)_{n}}\\
&= \frac{1}{\alpha-\beta}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q}{\alpha};q,\frac{-yq}{\alpha}\right).
\end{aligned}$$

Since $b_n = \alpha^{-1}a_{n-1}$,

$$\lim_{n\to\infty} b_n = \frac{\alpha^{-1}}{\alpha-\beta}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q}{\alpha};q,\frac{-yq}{\alpha}\right).$$

The sequence $\{c_n\}$ satisfies the same recurrence as $\{a_n\}$, with

$$c_0 = \frac{1-\beta}{\alpha-\beta}, \qquad c_1 = \frac{xq+yq+\alpha-\alpha\beta}{\alpha(\alpha-\beta)},$$

and thus, by reasoning similar to that above,

$$\begin{aligned}
\lim_{n\to\infty} c_n &= \frac{1}{q(\alpha-\beta)}\\
&\times\sum_{n=1}^{\infty}\frac{\left(\frac{y}{\alpha}\right)^{n-1} q^{n(n+1)/2}\left(\frac{-xq}{y\alpha}\right)_{n-1}\left(1-\beta+\frac{qx+qy}{\alpha}+(1-\beta)\left(\frac{xq}{\alpha}-\beta\right)\frac{q^n}{\alpha}\right)}{(q)_{n-1}(\beta q/\alpha)_n}\\
&= \frac{1-\beta}{(\alpha-\beta)}\sum_{n=0}^{\infty}\frac{(y/\alpha)^{n}q^{n(n+1)/2}(-xq/y\alpha)_{n}}{(q)_{n}(\beta q/\alpha)_{n}}\\
&\qquad + \frac{q(x+\beta y)}{(\alpha-\beta)(\alpha-\beta q)}\sum_{n=0}^{\infty}\frac{(yq/\alpha)^{n}q^{n(n+1)/2}(-xq/y\alpha)_{n}}{(q)_{n}(\beta q^2/\alpha)_{n}}\\
&= \frac{1-\beta}{\alpha-\beta}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q}{\alpha};q,\frac{-yq}{\alpha}\right)
\end{aligned}$$

$$+\frac{q(x+\beta y)}{(\alpha-\beta)(\alpha-\beta q)}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q^2}{\alpha};q,\frac{-yq^2}{\alpha}\right).$$

Also, $d_n = \alpha^{-1} c_{n-1}$, and so

$$\lim_{n\to\infty} d_n = \frac{\alpha^{-1}(1-\beta)}{\alpha-\beta}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q}{\alpha};q,\frac{-yq}{\alpha}\right) + \frac{\alpha^{-1}q(x+\beta y)}{(\alpha-\beta)(\alpha-\beta q)}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q^2}{\alpha};q,\frac{-yq^2}{\alpha}\right).$$

Thus,

$$\lim_{n\to\infty} G_n := \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix},$$

where

$$\begin{aligned}
g_{1,1} &= \frac{1}{\alpha-\beta}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q}{\alpha};q,\frac{-yq}{\alpha}\right),\\
g_{1,2} &= \frac{\alpha^{-1}}{\alpha-\beta}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q}{\alpha};q,\frac{-yq}{\alpha}\right),\\
g_{2,1} &= \frac{1-\beta}{\alpha-\beta}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q}{\alpha};q,\frac{-yq}{\alpha}\right)\\
&\qquad + \frac{q(x+\beta y)}{(\alpha-\beta)(\alpha-\beta q)}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q^2}{\alpha};q,\frac{-yq^2}{\alpha}\right)\\
g_{2,2} &= \frac{\alpha^{-1}(1-\beta)}{\alpha-\beta}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q}{\alpha};q,\frac{-yq}{\alpha}\right)\\
&\qquad + \frac{\alpha^{-1}q(x+\beta y)}{(\alpha-\beta)(\alpha-\beta q)}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q^2}{\alpha};q,\frac{-yq^2}{\alpha}\right).
\end{aligned}$$

From (4.6) H_n can be found from G_n by interchanging α and β, so that

$$\lim_{n\to\infty} H_n := \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix},$$

where

$$h_{1,1} = \frac{1}{\beta-\alpha}\,{}_1\phi_1\left(\frac{-xq}{y\beta};\frac{\alpha q}{\beta};q,\frac{-yq}{\beta}\right),$$
$$h_{1,2} = \frac{\beta^{-1}}{\beta-\alpha}\,{}_1\phi_1\left(\frac{-xq}{y\beta};\frac{\alpha q}{\beta};q,\frac{-yq}{\beta}\right),$$
$$h_{2,1} = \frac{1-\alpha}{\beta-\alpha}\,{}_1\phi_1\left(\frac{-xq}{y\beta};\frac{\alpha q}{\beta};q,\frac{-yq}{\beta}\right) + \frac{q(x+\alpha y)}{(\beta-\alpha)(\beta-\alpha q)}\,{}_1\phi_1\left(\frac{-xq}{y\beta};\frac{\alpha q^2}{\beta};q,\frac{-yq^2}{\beta}\right)$$
$$h_{2,2} = \frac{\beta^{-1}(1-\alpha)}{\beta-\alpha}\,{}_1\phi_1\left(\frac{-xq}{y\beta};\frac{\alpha q}{\beta};q,\frac{-yq}{\beta}\right) + \frac{\beta^{-1}q(x+\alpha y)}{(\beta-\alpha)(\beta-\alpha q)}\,{}_1\phi_1\left(\frac{-xq}{y\beta};\frac{\alpha q^2}{\beta};q,\frac{-yq^2}{\beta}\right).$$

Thus (4.7) gives

(4.12)

$$\begin{aligned}\lim_{n\to\infty} U_n M^{-n} &= \lim_{n\to\infty}(G_n\alpha^n + H_n\beta^n)M^{-n}\\ &= \lim_{n\to\infty}(G_n\alpha^n + H_n\beta^n)\begin{pmatrix}-\beta^{-1} & -\alpha^{-1}\\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha^{-n} & 0\\ 0 & \beta^{-n}\end{pmatrix}\begin{pmatrix}-\alpha\beta & -\beta\\ \alpha\beta & \alpha\end{pmatrix}\frac{1}{\alpha-\beta}\\ &= \begin{pmatrix}A & A'\\ B & B'\end{pmatrix}\frac{1}{\alpha-\beta},\end{aligned}$$

where

$$A = {}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q}{\alpha};q,\frac{-yq}{\alpha}\right) - {}_1\phi_1\left(\frac{-xq}{y\beta};\frac{\alpha q}{\beta};q,\frac{-yq}{\beta}\right)$$
$$A' = \frac{1}{\alpha}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q}{\alpha};q,\frac{-yq}{\alpha}\right) - \frac{1}{\beta}\,{}_1\phi_1\left(\frac{-xq}{y\beta};\frac{\alpha q}{\beta};q,\frac{-yq}{\beta}\right)$$
$$\begin{aligned}B = (1-\beta)\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q}{\alpha};q,\frac{-yq}{\alpha}\right) + \frac{q(x+\beta y)}{\alpha-\beta q}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q^2}{\alpha};q,\frac{-yq^2}{\alpha}\right)\\ -(1-\alpha)\,{}_1\phi_1\left(\frac{-xq}{y\beta};\frac{\alpha q}{\beta};q,\frac{-yq}{\beta}\right) - \frac{q(x+\alpha y)}{\beta-\alpha q}\,{}_1\phi_1\left(\frac{-xq}{y\beta};\frac{\alpha q^2}{\beta};q,\frac{-yq^2}{\beta}\right)\end{aligned}$$
$$\begin{aligned}B' = \frac{1-\beta}{\alpha}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q}{\alpha};q,\frac{-yq}{\alpha}\right) + \frac{q(x+\beta y)}{\alpha(\alpha-\beta q)}\,{}_1\phi_1\left(\frac{-xq}{y\alpha};\frac{\beta q^2}{\alpha};q,\frac{-yq^2}{\alpha}\right)\\ -\frac{1-\alpha}{\beta}\,{}_1\phi_1\left(\frac{-xq}{y\beta};\frac{\alpha q}{\beta};q,\frac{-yq}{\beta}\right) - \frac{q(x+\alpha y)}{\beta(\beta-\alpha q)}\,{}_1\phi_1\left(\frac{-xq}{y\beta};\frac{\alpha q^2}{\beta};q,\frac{-yq^2}{\beta}\right).\end{aligned}$$

If we let C_n and E_n denote the n-th numerator convergent and denominator convergent, respectively of the continued fraction

$$\frac{-\alpha\beta+xq}{\alpha+\beta+yq} + \frac{-\alpha\beta+xq^2}{\alpha+\beta+yq^2} + \frac{-\alpha\beta+xq^3}{\alpha+\beta+yq^3} + \cdots,$$

and define

$$V_n := \begin{pmatrix} C_n & C_{n-1} \\ E_n & E_{n-1} \end{pmatrix},$$

it can be seen that

$$V_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_{n+1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} U_{n+1}.$$

If the matrix F is defined by

$$\begin{aligned} F &:= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & A' \\ B & B' \end{pmatrix} M \frac{1}{\alpha-\beta} \\ &= \begin{pmatrix} (\alpha+\beta)(B-A)+\alpha\beta(A'-B') & B-A \\ (\alpha+\beta)A-\alpha\beta A' & A \end{pmatrix} \frac{1}{\alpha-\beta} \end{aligned} \tag{4.13}$$

and the linear fractional transformation $h(z)$ is as defined following (3.30), we get that

$$h(z) = \frac{\left(\frac{xq}{\alpha}-\beta\right) {}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq^2}{\alpha}\right) z - \left(\frac{xq}{\beta}-\alpha\right) {}_1\phi_1\left(\frac{-xq}{y\beta}; \frac{\alpha q}{\beta}; q, \frac{-yq^2}{\beta}\right)}{{}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq}{\alpha}\right) z - {}_1\phi_1\left(\frac{-xq}{y\beta}; \frac{\alpha q}{\beta}; q, \frac{-yq}{\beta}\right)}, \tag{4.14}$$

and (4.1) now follows in the case $|\alpha| = |\beta|$. (4.2) follows immediately from (4.1) and the remarks at the end of the introduction.

Note that we have used the elementary identity

$$\begin{aligned} \frac{q(x+\beta y)}{\alpha-\beta q} {}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q^2}{\alpha}; q, \frac{-yq^2}{\alpha}\right) - \beta\, {}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq}{\alpha}\right) \\ = \left(\frac{xq}{a}-\beta\right) {}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq^2}{\alpha}\right), \end{aligned}$$

and similarly with α and β interchanged.

Now assume that $|\alpha| \neq |\beta|$. First note that the difference equation

$$Y_n = (1+\lambda - zq^n)Y_{n+1} + (-\lambda + azq^n)Y_{n+2} \tag{4.15}$$

has a solution $Y_n = {}_1\phi_1(a; \lambda q; q, zq^n)$. (This can be checked simply by equating coefficients.) By Auric's theorem, see Corollary 11, Chapter IV of [19], this solution of (4.15) is minimal if $|\lambda| < 1$, and thus for $|\lambda| < 1$,

$$\frac{{}_1\phi_1(a; \lambda q; q, z)}{{}_1\phi_1(a; \lambda q; q, zq)} = 1+\lambda - z + \frac{-\lambda + az}{1+\lambda - zq} + \frac{-\lambda + azq}{1+\lambda - zq^2} + \cdots.$$

Putting $a = -\beta^{-1}xy^{-1}q$, $\lambda = \alpha/\beta$, and $z = -\beta^{-1}yq$, taking reciprocals, multiplying both sides by $-\alpha + xq/\beta$ and applying a simple equivalence transformation to the continued fraction, yields that for $|\alpha| < |\beta|$,

$$\frac{\left(\frac{xq}{\beta}-\alpha\right) {}_1\phi_1\left(\frac{-xq}{y\beta}; \frac{\alpha q}{\beta}; q, \frac{-yq^2}{\beta}\right)}{{}_1\phi_1\left(\frac{-xq}{y\beta}; \frac{\alpha q}{\beta}; q, \frac{-yq}{\beta}\right)} = \tag{4.16}$$

$$\frac{-\alpha\beta + xq}{\alpha + \beta + yq} + \frac{-\alpha\beta + xq^2}{\alpha + \beta + yq^2} + \frac{-\alpha\beta + xq^3}{\alpha + \beta + yq^3} + \cdots.$$

For $|\alpha| > |\beta|$, symmetry gives that

$$\frac{\left(\frac{xq}{\alpha} - \alpha\right) {}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq^2}{\alpha}\right)}{{}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq}{\alpha}\right)} = \frac{-\alpha\beta + xq}{\alpha + \beta + yq} + \frac{-\alpha\beta + xq^2}{\alpha + \beta + yq^2} + \frac{-\alpha\beta + xq^3}{\alpha + \beta + yq^3} + \cdots.$$

The conclusion follows by noting that for $|\alpha| < |\beta|$,

$$\frac{\left(\frac{xq}{\alpha} - \beta\right) {}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq^2}{\alpha}\right)\lambda^{n+1} - \left(\frac{xq}{\beta} - \alpha\right) {}_1\phi_1\left(\frac{-xq}{y\beta}; \frac{\alpha q}{\beta}; q, \frac{-yq^2}{\beta}\right)}{{}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq}{\alpha}\right)\lambda^{n+1} - {}_1\phi_1\left(\frac{-xq}{y\beta}; \frac{\alpha q}{\beta}; q, \frac{-yq}{\beta}\right)} \sim \frac{\left(\frac{xq}{\beta} - \alpha\right) {}_1\phi_1\left(\frac{-xq}{y\beta}; \frac{\alpha q}{\beta}; q, \frac{-yq^2}{\beta}\right)}{{}_1\phi_1\left(\frac{-xq}{y\beta}; \frac{\alpha q}{\beta}; q, \frac{-yq}{\beta}\right)},$$

while for $|\alpha| > |\beta|$,

$$\frac{\left(\frac{xq}{\alpha} - \beta\right) {}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq^2}{\alpha}\right)\lambda^{n+1} - \left(\frac{xq}{\beta} - \alpha\right) {}_1\phi_1\left(\frac{-xq}{y\beta}; \frac{\alpha q}{\beta}; q, \frac{-yq^2}{\beta}\right)}{{}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq}{\alpha}\right)\lambda^{n+1} - {}_1\phi_1\left(\frac{-xq}{y\beta}; \frac{\alpha q}{\beta}; q, \frac{-yq}{\beta}\right)} \sim \frac{\left(\frac{xq}{\alpha} - \beta\right) {}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq^2}{\alpha}\right)}{{}_1\phi_1\left(\frac{-xq}{y\alpha}; \frac{\beta q}{\alpha}; q, \frac{-yq}{\alpha}\right)}.$$

□

Consider the special case of the continued fraction in the theorem in which $x = 0$ and $y = 1$. Then

$$h(z) = \frac{-\beta\, {}_1\phi_1\left(0; \frac{\beta q}{\alpha}; q, \frac{-q^2}{\alpha}\right) z + \alpha\, {}_1\phi_1\left(0; \frac{\alpha q}{\beta}; q, \frac{-q^2}{\beta}\right)}{{}_1\phi_1\left(0; \frac{\beta q}{\alpha}; q, \frac{-q}{\alpha}\right) z - {}_1\phi_1\left(0; \frac{\alpha q}{\beta}; q, \frac{-q}{\beta}\right)}, \tag{4.17}$$

and thus that the sequential closure of the continued fraction

$$G(\alpha, \beta, q) := \frac{1}{1} - \frac{\alpha\beta}{\alpha + \beta + q} - \frac{\alpha\beta}{\alpha + \beta + q^2} - \frac{\alpha\beta}{\alpha + \beta + q^3} \cdots$$

is on the circle defined by

$$f(z) = \cfrac{1}{1 + \cfrac{-\beta\, {}_1\phi_1\left(0; \frac{\beta q}{\alpha}; q, \frac{-q^2}{\alpha}\right) z + \alpha\, {}_1\phi_1\left(0; \frac{\alpha q}{\beta}; q, \frac{-q^2}{\beta}\right)}{{}_1\phi_1\left(0; \frac{\beta q}{\alpha}; q, \frac{-q}{\alpha}\right) z - {}_1\phi_1\left(0; \frac{\alpha q}{\beta}; q, \frac{-q}{\beta}\right)}}.$$

Remark: Unless stated otherwise, we continue to restrict to the special case $x = 0$ and $y = 1$ for the remainder of this section, and continue to make use of the simplified expression for $h(z)$ given by (4.17).

Figure 1 shows the first 3500 approximants of $G(\exp(\imath\sqrt{7}), \exp(\imath\sqrt{5}), 0.1)$ and the corresponding circle $f(\mathbb{T})$ predicted by the theory. The larger dots show the points, again predicted by the theory, of highest and lowest concentration of approximants. Note that the error, $\varepsilon_n = O(10^{-n})$ and experimentally, $\min_{z\in\mathbb{T}} |A_n/B_n - f(z)| \approx 10^{-n}$ in agreement with the theory.

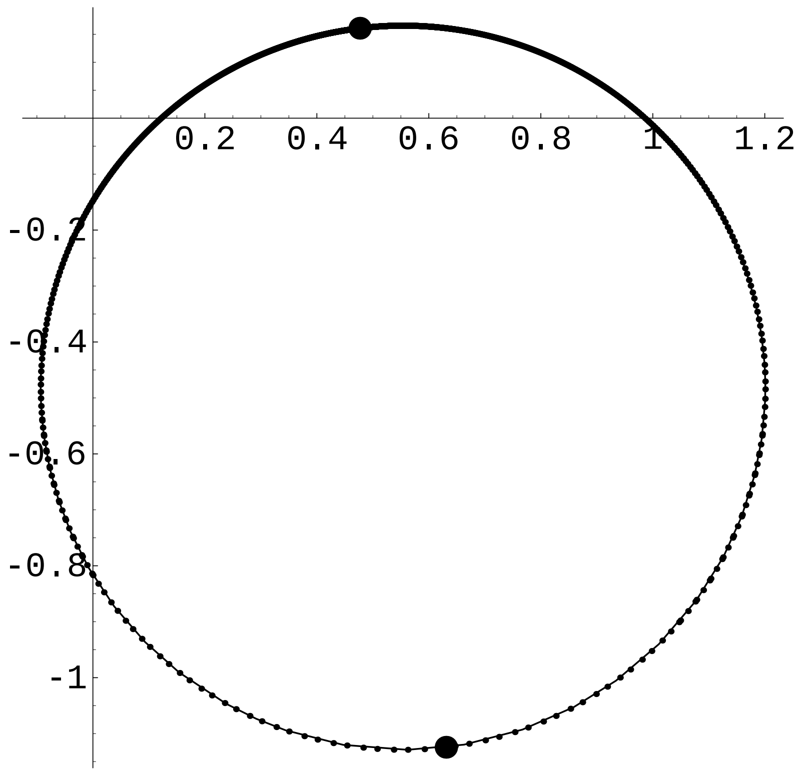

FIGURE 1. The convergence of $G(\exp(\imath\sqrt{7}), \exp(\imath\sqrt{5}), 0.1)$

Figure 2 shows the first 2700 approximants of $G(\exp(\imath\sqrt{7}), \exp(\imath(\sqrt{7} + 2\pi/11)), 0.1)$ and its convergence to the eleven limit points $f(2k\pi/11)$, where $f(z)$ is the associated linear fractional transformation, together with part of the circle $f(\mathbb{T})$. The error is in agreement with theory: $|A_n/B_n - f(2(n+1)\pi/11))| \approx 10^{-n}$. This rapid convergence is the reason that the graph appears to show only twelve approximants (the zeroth approximant is a little removed from all of the limit points).

It was shown in the introduction that the sequence of approximants of the continued fraction $3/2 + K_{n=1}^{\infty} \frac{-1}{3/2}$ is dense in $\mathbb{R}$. A natural question is: what is the point of highest concentration of approximants? We can now

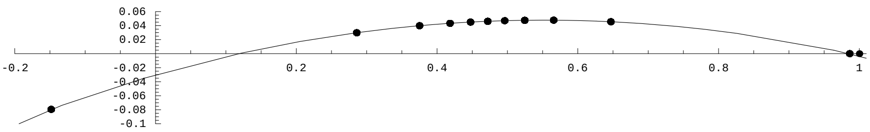

FIGURE 2. The convergence of $G(\exp(\imath\sqrt{7}), \exp(\imath(\sqrt{7}+2\pi/11)), 0.1)$

answer this question. We can view the continued fraction $3/2 + K_{n=1}^{\infty} \frac{-1}{3/2}$ as

$$\alpha + \beta + K_{n=1}^{\infty} \frac{-\alpha\beta}{\alpha+\beta+q^n},$$

with $\alpha = 3/4 + i\sqrt{7}/4$, $\beta = 3/4 - i\sqrt{7}/4$ and $q = 0$. By Theorem 10

$$s.c. \left(\alpha + \beta + K_{n=1}^{\infty} \frac{-\alpha\beta}{\alpha+\beta+q^n}\right) = \alpha + \beta - \frac{\beta\mathbb{T}_\infty - \alpha}{\mathbb{T}_\infty - 1} = \frac{\alpha\mathbb{T}_\infty - \beta}{\mathbb{T}_\infty - 1}.$$

Thus $a = \alpha$, $b = -\beta$, $c = 1$, and $d = -1$. From Corollary 5, the point of highest concentration of approximants is

$$\frac{1}{2}\left(\frac{a}{c} + \frac{b}{d}\right) = \frac{1}{2}\left(\frac{\alpha}{1} + \frac{-\beta}{-1}\right) = \frac{\alpha+\beta}{2} = \frac{3}{4},$$

and the limiting probability density function is:

$$p(x) = \frac{\sqrt{7}}{2\pi(2x^2 - 3x + 2)}.$$

Figure 3 shows the distribution of the first 3000 approximants of $3/2 + K_{n=1}^{\infty} \frac{-1}{3/2}$ (with about 300 extreme values omitted and scaled to have area equal to 1), together with the point $x = 3/4$ of predicted highest concentration and the limiting probability density function $p(x) = \sqrt{7}/(2\pi(2x^2 - 3x + 2))$. Once again, theory and experiment are in complete agreement.

Corollary 6 is now applied to obtain convergent continued fractions. This corollary could be applied to the more general continued fraction in Theorem 10, but for the sake of simplicity and ease of notation we restrict once again to the special case where $x = 0$ and $y = 1$. We also revert to series notation for ease of understanding.

Corollary 10. *Let $|q| < 1$ and let α and β be distinct points on the unit circle such that α/β is not a root of unity.*
(i) Set

$$\omega_n = -\frac{\alpha^n - \beta^n}{\alpha^{n-1} - \beta^{n-1}}.$$

Then

$$(4.18) \quad \frac{-\alpha\beta}{\alpha+\beta+q} + \frac{-\alpha\beta}{\alpha+\beta+q^2} + \frac{-\alpha\beta}{\alpha+\beta+q^3+\omega_3} + \frac{-\omega_3 q^4}{\alpha+\beta+q^4+\omega_4} + K_{n=5}^{\infty} \frac{-q\alpha\beta\,\omega_{n-1}/\omega_{n-2}}{q^n - \omega_{n-1}q + \alpha + \beta + \omega_n}$$

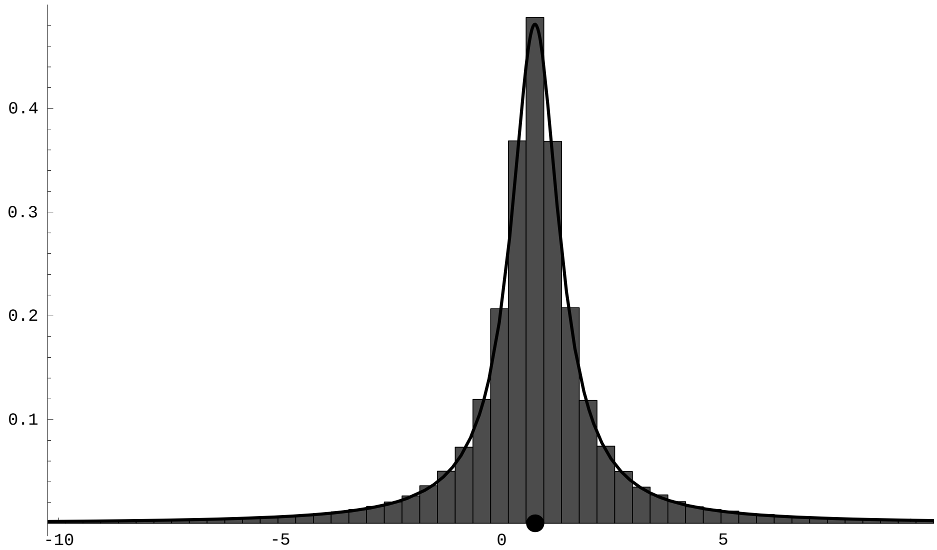

FIGURE 3. The distribution of the first 3000 approximants of $3/2 + K_{n=1}^{\infty} \frac{-1}{3/2}$, with the point $x = 3/4$ of predicted highest concentration and the limiting probability density function $p(x) = \sqrt{7}/(2\pi(2x^2 - 3x + 2))$.

$$= -\alpha\beta \frac{\displaystyle\sum_{n=0}^{\infty} \frac{\alpha^{-n} q^{n(n+3)/2}}{(q;q)_n (\beta q/\alpha;q)_n} - \sum_{n=0}^{\infty} \frac{\beta^{-n} q^{n(n+3)/2}}{(q;q)_n (\alpha q/\beta;q)_n}}{\displaystyle\alpha \sum_{n=0}^{\infty} \frac{\alpha^{-n} q^{n(n+1)/2}}{(q;q)_n (\beta q/\alpha;q)_n} - \beta \sum_{n=0}^{\infty} \frac{\beta^{-n} q^{n(n+1)/2}}{(q;q)_n (\alpha q/\beta;q)_n}}.$$

(ii) Set

$$\omega_n = -\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n}.$$

Then

$$\frac{-\alpha\beta}{\alpha+\beta+q} + \frac{-\alpha\beta}{\alpha+\beta+q^2} + \frac{-\alpha\beta}{\alpha+\beta+q^3+\omega_3} + \frac{-\omega_3 q^4}{\alpha+\beta+q^4+\omega_4} + K_{n=5}^{\infty} \frac{-q\alpha\beta\,\omega_{n-1}/\omega_{n-2}}{q^n - \omega_{n-1}q + \alpha + \beta + \omega_n} \tag{4.19}$$

$$= -\frac{\displaystyle\beta \sum_{n=0}^{\infty} \frac{\alpha^{-n} q^{n(n+3)/2}}{(q;q)_n (\beta q/\alpha;q)_n} - \alpha \sum_{n=0}^{\infty} \frac{\beta^{-n} q^{n(n+3)/2}}{(q;q)_n (\alpha q/\beta;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{\alpha^{-n} q^{n(n+1)/2}}{(q;q)_n (\beta q/\alpha;q)_n} - \sum_{n=0}^{\infty} \frac{\beta^{-n} q^{n(n+1)/2}}{(q;q)_n (\alpha q/\beta;q)_n}}.$$

(iii)

$$-\beta+\frac{\beta q}{\alpha+q}\;+\;K_{n=2}^{\infty}\frac{-\alpha\beta q}{q^n+\alpha+\beta q}\;=\;-\beta\frac{\displaystyle\sum_{n=0}^{\infty}\frac{\alpha^{-n}q^{n(n+3)/2}}{(q;q)_n(\beta q/\alpha;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{\alpha^{-n}q^{n(n+1)/2}}{(q;q)_n(\beta q/\alpha;q)_n}}. \tag{4.20}$$

Proof. (i) In Corollary 6, let $k=0$, $q_n=0$ and $p_n=q^n$, and then, for $n\geq 5$, c_n simplifies to $-q\alpha\beta\omega_{n-1}/\omega_{n-2}$ and d_n simplifies to $\alpha+\beta+q^n+\omega_n-\omega_{n-1}q$. The fourth partial numerator similarly simplifies to $-\omega_3 q^4$. Thus this continued fraction converges to $h(\lambda)$ and it can be seen from (4.17) that $h(\lambda)$ has the valued claimed for the limit of the continued fraction.

The proof of (ii) is similar, except we take $k=-1$ in Corollary 6 and noting from (4.17) that $h(1)$ has the valued claimed for the limit of the continued fraction.

Part (iii) follows from (3.37), after noting from (4.17) that $h(\infty)$ has the valued claimed for the limit of the continued fraction. □

In some cases the infinite series in the theorem above can be expressed as infinite products.

Corollary 11. *Let* $|q|<1$. *Then*

$$1-\frac{q}{1+q}+K_{n=2}^{\infty}\frac{q^2}{1-q^2+q^{2n-1}}=\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{4.21}$$

Proof. In (4.20), replace q by q^2, set $\beta=-q$ and $\alpha=q$ and simplify the resulting continued fraction by applying a sequence of similarity transformations.

For the right side we use two identities due to Rogers [25] (see also [30] and [29], identities **A.16** and **A.20**):

$$\sum_{n=0}^{\infty}\frac{q^{n(n+2)}}{(q^4;q^4)_n}=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty(-q^2;q^2)_\infty},$$
$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4;q^4)_n}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty(-q^2;q^2)_\infty}.$$

Finally, cancel a factor of q on each side □

Remark: The continued fraction above is clearly a transformed version of the Rogers-Ramanujan continued fraction which converges to the same limit as the original continued fraction.

4.1. **Example: analytic behavior of $K\frac{-1}{a+q^n}$ for $|q|<1$.** The results of this section provide the opportunity to examine in detail, for $|q|<1$, the relationship between the continued fraction

$$K\frac{-1}{a+q^n} \tag{4.22}$$

and its series equivalents. We begin with some remarks about the convergence of the continued fraction (4.22) for $|q| < 1$.

First of all, for $|q| < 1$, (4.22) is a limit periodic continued fraction. Now, depending on the complex value of a, (4.22) can be of either parabolic, elliptic, or loxodromic type, see [19] for the definitions. More specifically, it is easy to check that (4.22) is of parabolic type if and only if $a = \pm 2$. Theorem 32 (with $p = 0$) of chapter III of [19] gives that in this case the continued fraction converges. In fact we will give the limit of the continued fraction when $a = \pm 2$ below.

A more involved computation yields that (4.22) is of elliptic type if and only if a is both real and satisfies $-2 < a < 2$. It is easy to see that this is equivalent to $a = \omega + \overline{\omega}$ with ω on the unit circle and $\omega \neq \pm 1$, which is exactly the $\beta = \overline{\alpha}$ case of Theorem 10. The continued fraction (4.22) is loxodromic in all other cases and hence converges by Theorem 28, chapter III of [19]. This is, of course, the same as saying that its sequential closure consists of a single point.

Note that the $|\alpha| \neq |\beta|$ case of Theorem 10 is in agreement with remark (v) following Theorem 7. In this case we are able to conclude that the constants a, b, c, and d still make sense. Putting $\beta = \alpha^{-1}$ in Theorem 10 thus gives detailed asymptotics for the approximants when $a \neq 2$.

We now compute the limit of $K\frac{-1}{a+q^n}$ in the loxodromic and parabolic cases. Define

$$H(a', b, c, d, q) := \frac{1}{1} + \frac{-a'b + cq}{a' + b + dq} + \frac{-a'b + cq^2}{a' + b + dq^2} + \cdots + \frac{-a'b + cq^n}{a' + b + dq^n} + \cdots.$$

Let A_n denote the n-th numerator convergent of this continued fraction and let B_n denote its n-th denominator convergent.

In Theorem 2.2 in [8], it was shown that if $|a'/b| < 1$ and $|q| < 1$, then

$$\lim_{N\to\infty} \frac{A_N}{b^{N-1}} = \sum_{n=0}^{\infty} \frac{(d/b)^n q^{n(n+1)/2}(-cq/db)_n}{(a'/b)_{n+1}(q)_n}, \tag{4.23}$$

$$\lim_{N\to\infty} \frac{B_N - A_N}{b^{N-1}} = (cq/b - a') \sum_{n=0}^{\infty} \frac{(d/b)^n q^{n(n+3)/2}(-cq/db)_n}{(a'/b)_{n+1}(q)_n},$$

and thus that

$$\frac{1}{H(a', b, c, d, q)} - 1 = \frac{(cq/b - a') \displaystyle\sum_{n=0}^{\infty} \frac{(d/b)^n q^{n(n+3)/2}(-cq/db)_n}{(a'/b)_{n+1}(q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{(d/b)^n q^{n(n+1)/2}(-cq/db)_n}{(a'/b)_{n+1}(q)_n}}.$$

Notice that this equation is the same as (4.16). Indeed, we could have finished the proof of Theorem 10 using these results from [8], but the self-contained method used seemed preferable.

Let $|a|>2$ and set $b=(a+\sqrt{a^2-4})/2$ if $a>2$ and $b=(a-\sqrt{a^2-4})/2$ if $a<-2$, $a'=1/b$, $d=1$ and $c=0$. It is immediate that

$$K_{n=1}^{\infty}\frac{-1}{a+q^n}=-\frac{1}{b}\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(1/b)^n q^{n(n+3)/2}}{(1/b^2)_{n+1}(q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(1/b)^n q^{n(n+1)/2}}{(1/b^2)_{n+1}(q)_n}}=-\frac{1}{b}\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(1/b)^n q^{n(n+3)/2}}{(q/b^2)_{n}(q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(1/b)^n q^{n(n+1)/2}}{(q/b^2)_{n}(q)_n}}.$$

This gives the limit of the continued fraction in the loxodromic case.

To compute the limit in the parabolic case, we need the following result.

Lemma 1. *If $s(n)=\sum_{k\geq 0}f_k(n)$ is a finite sum (or a convergent series) for each n, $\lim_{n\to\infty}f_k(n)=f_k$, $|f_k(n)|\leq M_k$, and $\sum_{k=0}^{\infty}M_k<\infty$, then*

$$\lim_{n\to\infty}s(n)=\sum_{k=0}^{\infty}f_k.$$

This result follows as a consequence of the Weierstrass M-test and is also known as *Tannery's Theorem* (see [11], for example).

Let

$$G(n):=K_{k=1}^{\infty}\frac{-1}{2+1/(n^2+n)+q^k}$$

Let $C_{k,n}$ denote the k-th numerator convergent of $G(n)$ and let $D_{k,n}$ denote its k-th numerator convergent. From (4.23), with $b=(n+1)/n$, $a'=1/b$, $d=1$ and $c=0$,

$$C_n:=\lim_{k\to\infty}\frac{C_{k,n}}{(1+1/n)^{k-1}}=-\frac{1}{1+1/n}\sum_{n=0}^{\infty}\frac{(n/(n+1))^k q^{k(k+3)/2}}{(n^2/(n+1)^2)_{k+1}(q)_k},$$

$$D_n:=\lim_{k\to\infty}\frac{D_{k,n}}{(1+1/n)^{k-1}}=\sum_{n=0}^{\infty}\frac{(n/(n+1))^k q^{k(k+1)/2}}{(n^2/(n+1)^2)_{k+1}(q)_k}.$$

Define

$$s_1(n)=\frac{1+2n}{(1+n)^2}C_n,$$

$$s_2(n)=\frac{1+2n}{(1+n)^2}D_n.$$

Next,

$$K_{k=1}^{\infty}\frac{-1}{2+q^k}=\lim_{n\to\infty}K_{k=1}^{\infty}\frac{-1}{2+1/(n^2+n)+q^k}=\lim_{n\to\infty}\frac{C_n}{D_n}=\lim_{n\to\infty}\frac{s_1(n)}{s_2(n)}.$$

Since $s_1(n)=\sum_{k\geq 0}f_k(n)$, where

$$f_k(n)=-\frac{1}{1+1/n}\frac{(n/(n+1))^k q^{k(k+3)/2}}{(n^2q/(n+1)^2;q)_k(q;q)_k}$$

and

$$\lim_{n\to\infty} f_k(n) = -\frac{q^{k(k+3)/2}}{(q;q)_k^2} =: f_k,$$
$$|f_k(n)| \le 2\frac{|q|^{k(k+3)/2}}{(|q|;|q|)_k^2} =: M_k,$$
$$\sum_{k\ge 0} M_k = \sum_{k\ge 0} 2\frac{|q|^{k(k+3)/2}}{(|q|;|q|)_k^2} < \infty,$$

it follows that

$$\lim_{n\to\infty} s_1(n) = \sum_{k\ge 0} f_k = -\sum_{k\ge 0}\frac{q^{k(k+3)/2}}{(q;q)_k^2}.$$

Likewise

$$\lim_{n\to\infty} s_2(n) = \sum_{k\ge 0}\frac{q^{k(k+1)/2}}{(q;q)_k^2},$$

and so

$$K_{n=1}^{\infty}\frac{-1}{2+q^n} = -\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n+3)/2}}{(q;q)_n^2}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q)_n^2}}.$$

By a similar argument, we get that

$$K_{n=1}^{\infty}\frac{-1}{-2+q^n} = \frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+3)/2}}{(q;q)_n^2}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n^2}}.$$

5. Applications to (r, s)-matrix continued fractions

In [18], the authors define a generalization of continued fractions called (r, s)-matrix continued fractions. This generalization unifies a number of generalizations of continued fractions including "generalized (vector valued) continued fractions" and "G-continued fractions", see [19] for terminology.

Here we show that our results apply to limit periodic (r, s)-matrix continued fractions with eigenvalues of equal magnitude, giving estimates for the asymptotics of their approximants so that their sequential closures can be determined.

For consistency we closely follow the notation used in [18] to define (r, s)-matrix continued fractions. Let $M_{s,r}(\mathbb{C})$ denote the set of $s \times r$ matrices over the complex numbers. Let θ_k be a sequence of $n \times n$ matrices over $\mathbb{C}$. Assume that $r + s = n$. A (r, s)-matrix continued fraction is associated with a recurrence system of the form $Y_k = Y_{k-1}\theta_k$. The continued fraction is defined by its sequence of approximants. These are sequences of $s \times r$ matrices defined in the following manner.

Define the function $f : D \in M_n(\mathbb{C}) \rightarrow M_{s,r}(\mathbb{C})$ by

$$f(D) = B^{-1}A, \tag{5.1}$$

where B is the $s \times s$ submatrix consisting of the last s elements from both the rows and columns of D, and A is the $s \times r$ submatrix consisting of the first r elements from the last s rows of D.

Then the k-th approximant of the (r, s)-matrix continued fraction associated with the sequence θ_k is defined to be

$$s_k := f(\theta_k \theta_{k-1} \cdots \theta_2 \theta_1). \tag{5.2}$$

To apply Theorem 4 to this situation, we endow $M_{s\times r}(\mathbb{C})$ with a metric by letting the distance function for two such matrices be the maximum absolute value of the respective differences of corresponding pairs of elements. Then, providing that the f is continuous, our theorem can be applied. (Note that f will be continuous providing that it exists, since the inverse function of a matrix is continuous when it exists.)

Let $\lim_{k\rightarrow\infty} \theta_k = \theta$, for some $\theta \in M_n(\mathbb{C})$. Then the recurrence system is said to be of Poincaré type and the (r, s)-matrix continued fraction is called limit periodic.

After this definition Theorem 4 can be applied and the following theorem results.

Theorem 11. *Suppose that the condition $\sum_{k\geq 1} ||\theta_k - \theta|| < \infty$ holds, that the matrix θ is diagonalizable, and that the eigenvalues of θ are all of magnitude 1. Then the kth approximant s_k has the asymptotic formula*

$$s_k \sim f(\theta^k F), \tag{5.3}$$

where F is the matrix defined by the convergent product

$$F := \lim_{k\rightarrow\infty} \theta^{-k} \theta_k \theta_{k-1} \cdots \theta_2 \theta_1.$$

Note that because of the way that (r, s)-matrix continued fractions are defined, we have taken products in the reverse order than the rest of the paper.

As a consequence of this asymptotic, the sequential closure can be determined from

$$s.c.(s_k) = s.c.(f(\theta^k F)).$$

In one general case, detailed in the following theorem, we actually get a convergence theorem.

Theorem 12. *Let θ_k be a sequence of $n \times n$ matrices over $\mathbb{C}$ satisfying*

$$\sum_{k\geq 1} ||\theta_k - \theta|| < \infty,$$

where θ is a diagonal (or antidiagonal) matrix with all diagonal (or antidiagonal) elements of absolute value 1. Let r and s be positive integers with $r + s = n$.

Then the matrix

$$F := \lim_{k\to\infty} \theta^{-k}\theta_k\theta_{k-1}\cdots\theta_2\theta_1$$

exists. Suppose further that the bottom right $s \times s$ submatrix of F is nonsingular. Then the (r, s)-matrix continued fraction defined by equation (5.2) converges to $f(F)$. If θ is antidiagonal, then the even approximants of (r, s)-matrix continued fraction defined by equation (5.2) tend to $f(F)$, while its odd approximants tend to $f(AF)$, where A is the antidiagonal matrix with 1s along its antidiagonal.

Proof. The matrix F exists by Theorem 4 (or more precisely, the "transposed" version of Theorem 4). Let

$$\theta = diag(\lambda_1, \ldots, \lambda_n).$$

By (5.3),

$$\begin{aligned} s_k &\sim f(\theta^k F) \\ &= \left(\begin{pmatrix} \lambda_{n-s+1}^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^k \end{pmatrix} \begin{pmatrix} F_{n-s+1,n-s+1} & \cdots & F_{n-s+1,n} \\ \vdots & \ddots & \vdots \\ F_{n,n-s+1} & \cdots & F_{n,n} \end{pmatrix} \right)^{-1} \\ &\qquad \times \begin{pmatrix} \lambda_{n-s+1}^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^k \end{pmatrix} \begin{pmatrix} F_{n-s+1,1} & \cdots & F_{n-s+1,r} \\ \vdots & \ddots & \vdots \\ F_{n,1} & \cdots & F_{n,r} \end{pmatrix} \\ &= \begin{pmatrix} F_{n-s+1,n-s+1} & \cdots & F_{n-s+1,n} \\ \vdots & \ddots & \vdots \\ F_{n,n-s+1} & \cdots & F_{n,n} \end{pmatrix}^{-1} \begin{pmatrix} F_{n-s+1,1} & \cdots & F_{n-s+1,r} \\ \vdots & \ddots & \vdots \\ F_{n,1} & \cdots & F_{n,r} \end{pmatrix} \\ &= f(F). \end{aligned}$$

Thus s_k converges to the final matrix product above.

For the case where θ is an antidiagonal matrix, θ^{2k} is a diagonal matrix and the proof for the even approximants is virtually the same as for the case where θ is a diagonal matrix. If θ is an antidiagonal matrix, θ^{2k+1} is also an antidiagonal matrix. Once again by (5.3),

$$\begin{aligned}
s_{2k+1} &\sim f(\theta^{2k+1}F)\\
&= \left(\begin{pmatrix} 0 & \dots & (\theta^{2k+1})_{n-s+1,s} \\ \vdots & \iddots & \vdots \\ (\theta^{2k+1})_{n,1} & \dots & 0 \end{pmatrix} \begin{pmatrix} F_{1,n-s+1} & \dots & F_{1,n} \\ \vdots & \ddots & \vdots \\ F_{s,n-s+1} & \dots & F_{s,n} \end{pmatrix} \right)^{-1} \\
&\qquad \times \begin{pmatrix} 0 & \dots & (\theta^{2k+1})_{n-s+1,s} \\ \vdots & \iddots & \vdots \\ (\theta^{2k+1})_{n,1} & \dots & 0 \end{pmatrix} \begin{pmatrix} F_{1,1} & \dots & F_{1,r} \\ \vdots & \ddots & \vdots \\ F_{s,1} & \dots & F_{s,r} \end{pmatrix} \\
&= \left(\begin{pmatrix} 0 & \dots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \dots & 0 \end{pmatrix} \begin{pmatrix} F_{1,n-s+1} & \dots & F_{1,n} \\ \vdots & \ddots & \vdots \\ F_{s,n-s+1} & \dots & F_{s,n} \end{pmatrix} \right)^{-1} \\
&\qquad\qquad \times \begin{pmatrix} 0 & \dots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \dots & 0 \end{pmatrix} \begin{pmatrix} F_{1,1} & \dots & F_{1,r} \\ \vdots & \ddots & \vdots \\ F_{s,1} & \dots & F_{s,r} \end{pmatrix} \\
&= f(AF),
\end{aligned}$$

where A is the antidiagonal matrix with 1's along the antidiagonal. Thus s_{2k+1} converges to the final matrix product above. □

With additional information, this last conclusion can often be strengthened, for example, to $s.c.(s_n) = h(s.c.(\theta^k F))$, as in the case of continued fractions studied in section 3. The computation of $s.c.(\theta^k F)$ can then be accomplished through Pontryagin duality.

Consider now the $n = 2$ antidiagonal case of Theorem 12. The matrix θ then has the form

$$\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Choose θ_k to have the form

$$\theta_k = \begin{pmatrix} b_k & 1 \\ 1 + a_k & 0 \end{pmatrix}.$$

Using the correspondence between matrices and continued fractions (3.1), we at once obtain the following corollary, first given in [7].

Corollary 12. *Let the sequences* $\{a_n\}$ *and* $\{b_n\}$ *satisfy* $a_n \neq -1$ *for* $n \geq 1$, $\sum |a_n| < \infty$ *and* $\sum |b_n| < \infty$. *Then*

$$b_0 + K_{n=1}^{\infty} \frac{1 + a_n}{b_n}$$

diverges. In fact, for $p = 0, 1$,

$$\lim_{n\to\infty} P_{2n+p} = A_p \neq \infty, \qquad \lim_{n\to\infty} Q_{2n+p} = B_p \neq \infty,$$

and

$$A_1B_0 - A_0B_1 = \prod_{n=1}^{\infty}(1 + a_n).$$

In fact, Corollary 12 is also the $\alpha = 1$, $\beta = -1$ (so $m = 2$), $q_n = a_n$ and $p_n = b_n$ case of Corollary 7. When $a_n = 0$, this corollary reduces to the famous Stern-Stolz theorem discussed in the introduction.

One of the main results of the paper [7] was Corollary 7, which we applied to obtain an infinite sequence of theorems, similar to the Stern-Stolz theorem, but with continued fractions of different ranks. Notice that Theorem 12 provides yet another family of generalizations.

It is interesting to compare Corollary 12 with the "The General Stern-Stolz Theorem" from [3] in the case of continued fractions. The corollary for the case of complex continued fractions is:

Corollary 13. *[Corollary 7.5 of* [3]*] If* $\sum_n |1 - |a_n||$ *and* $\sum_n |b_n|$ *converge, then* $K(a_n|b_n)$ *is strongly divergent.*

The first condition in this result is weaker than analogous condition in Corollary 12 above. But it should be remarked that that Theorem 1, Corollary 12, and Corollary 13 are, in fact, equivalent. In fact, the two corollaries follow from Theorem 1 by an equivalence transformation (and a little analysis). Next, the condition on the partial numerators in Corollary 13 encodes the information that the matrices representing the continued fraction are a perturbation of unitary matrices. We could have obtained the same result by using Theorem 3, however in this situation one does not obtain as detailed information about the limits of the convergents. In particular, Corollary 12 also proves the convergence of the subsequences of convergents $\{P_n\}$ and $\{Q_n\}$ of equal parity. Corollary 13 does not furnish this part of the conclusion. On the other hand, it does prove strong divergence, defined in section 2. Indeed, the continued fraction in Corollary 13 is not necessarily limit peoridic.

6. Poincaré type recurrence relations with characteristic roots on the unit circle

Let the sequence $\{x_n\}_{n\geq 0}$ have the initial values $x_0, \ldots, x_{p-1}$ and be subsequently defined by

$$x_{n+p} = \sum_{r=0}^{p-1} a_{n,r}x_{n+r}, \tag{6.1}$$

for $n \geq 0$. Suppose also that there are numbers $a_0, \ldots, a_{p-1}$ such that

$$\lim_{n\to\infty} a_{n,r} = a_r, \qquad 0 \leq r \leq p-1. \tag{6.2}$$

A recurrence of the form (6.1) satisfying the condition (6.2) is called a Poincaré-type recurrence, (6.2) being known as the Poincaré condition. Such recurrences were initially studied by Poincaré who proved that if the roots of the characteristic equation

$$t^p - a_{p-1}t^{p-1} - a_{p-2}t^{p-2} - \cdots - a_0 = 0 \tag{6.3}$$

have distinct norms, then the ratios of consecutive terms in the recurrence (for any set of initial conditions) tend to one of the roots. See [22]. Because the roots are also the eigenvalues of the associated companion matrix, they are also referred to as the eigenvalues of (6.1). This result was improved by O. Perron, who obtained a number of theorems about the limiting asymptotics of such recurrence sequences. Perron [21] made a significant advance in 1921 when he proved the following theorem which for the first time treated cases of eigenvalues which repeat or are of equal norm.

Theorem 13. *Let the sequence $\{x_n\}_{n\geq 0}$ be defined by initial values x_0, $\dots$, x_{p-1} and by (6.1) for $n \geq 0$. Suppose also that there are numbers $a_0, \dots, a_{p-1}$ satisfying (6.2). Let q_1, $q_2, \dots q_\sigma$ be the distinct moduli of the roots of the characteristic equation (6.3) and let l_λ be the number of roots whose modulus is q_λ, multiple roots counted according to multiplicity, so that*

$$l_1 + l_2 + \dots l_\sigma = p.$$

Then, provided $a_{n,0}$ be different from zero for $n \geq 0$, the difference equation (6.1) *has a fundamental system of solutions, which fall into σ classes, such that, for the solutions of the λ-th class and their linear combinations,*

$$\limsup_{n\to\infty} \sqrt[n]{|x_n|} = q_\lambda.$$

The number of solutions of the λ-th class is l_λ.

Thus when all of the characteristic roots have norm 1, this theorem gives that

$$\limsup_{n\to\infty} \sqrt[n]{|x_n|} = 1.$$

Another related paper is [17] where the authors study products of matrices and give a sufficient condition for their boundedness. This is then used to study "equimodular" limit periodic continued fractions, which are limit periodic continued fractions in which the characteristic roots of the associated 2×2 matrices are all equal in modulus. The matrix theorem in [17] can also be used to obtain results about the boundedness of recurrence sequences. We study a more specialized situation here and obtain far more detailed information as a consequence.

Our focus is on the case where the characteristic roots are distinct numbers on the unit circle. Under a condition stronger than (6.2) we will show that all non-trivial solutions of such recurrences are asymptotic to a linear recurrence with constant coefficients. Specifically, our theorem is:

Theorem 14. *Let the sequence* $\{x_n\}_{n\geq 0}$ *be defined by initial values* x_0, $\ldots$, x_{p-1} *and by* (6.1) *for* $n \geq 0$. *Suppose also that there are numbers* $a_0, \ldots, a_{p-1}$ *such that*

$$\sum_{n=0}^{\infty} |a_r - a_{n,r}| < \infty, \qquad 0 \leq r \leq p-1.$$

Put

$$\varepsilon_n = \max_{0\leq r<p} \left(\sum_{i>n} |a_r - a_{i,r}| \right).$$

Suppose further that the roots of the characteristic equation

$$t^p - a_{p-1}t^{p-1} - a_{p-2}t^{p-2} - \cdots - a_0 = 0 \tag{6.4}$$

are distinct and all on the unit circle, with values α_0, $\ldots$, α_{p-1}. *Then there exist complex numbers* $c_0, \ldots, c_{p-1}$ *such that*

$$\left| x_n - \sum_{i=0}^{p-1} c_i \alpha_i^n \right| = O\left(\varepsilon_n\right). \tag{6.5}$$

Proof. Define

$$M := \begin{pmatrix} a_{p-1} & a_{p-2} & \ldots & a_1 & a_0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

By the correspondence between polynomials and companion matrices, the eigenvalues of M are $\alpha_1, \ldots, \alpha_p$, so that M is diagonalizable. For $n \geq 1$, define

$$D_n := \begin{pmatrix} a_{n-1,p-1} & a_{n-1,p-2} & \ldots & a_{n-1,1} & a_{n-1,0} \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

Thus the matrices M and D_n satisfy the conditions of Theorem 4. From (6.1) it follows that

$$\begin{pmatrix} x_{n+p-1} \\ x_{n+p-2} \\ \vdots \\ x_n \end{pmatrix} = \prod_{j=1}^{n} D_j \begin{pmatrix} x_{p-1} \\ x_{p-2} \\ \vdots \\ x_0 \end{pmatrix}.$$

Let F have the same meaning as in Theorem 4. Part (i) then gives that

$$\left\| \begin{pmatrix} x_{n+p-1} \\ x_{n+p-2} \\ \vdots \\ x_n \end{pmatrix} - F\,M^n \begin{pmatrix} x_{p-1} \\ x_{p-2} \\ \vdots \\ x_0 \end{pmatrix} \right\| = O\left(\varepsilon_n\right).$$

(6.5) follows immediately by considering the bottom entry on the left side. This completes the proof. □

The following corollary, proved in [7], is immediate.

Corollary 14. *Let the sequence $\{x_n\}_{n\geq 0}$ be defined by initial values x_0, $\ldots$, x_{p-1} as well as (6.1) for $n \geq 0$. Suppose also that there are numbers $a_0, \ldots, a_{p-1}$ such that*

$$\sum_{n=0}^{\infty} |a_r - a_{n,r}| < \infty, \qquad 0 \leq r \leq p-1.$$

Assume that the roots of the characteristic equation

$$t^p - a_{p-1}t^{p-1} - a_{p-2}t^{p-2} - \cdots - a_0 = 0$$

are distinct roots of unity α_0, $\ldots$, α_{p-1}. Let m be the least positive integer such that, for all $j \in \{0, 1, \ldots, p-1\}$, $\alpha_j^m = 1$. Then, for $0 \leq j \leq m-1$, the subsequence $\{x_{mn+j}\}_{n=0}^{\infty}$ converges. Set $l_j = \lim_{n\to\infty} x_{nm+j}$, for integers $j \geq 0$. Then the (periodic) sequence $\{l_j\}$ satisfies the recurrence relation

$$l_{n+p} = \sum_{r=0}^{p-1} a_r l_{n+r},$$

and thus there exist constants $c_0, \cdots, c_{p-1}$ such that

$$l_n = \sum_{i=0}^{p-1} c_i \alpha_i^n.$$

7. Conclusion

We have studied convergent subsequences of approximants of complex continued fractions and generalizations. There is an interesting pattern of relationships between the limits and asymptotics of subsequences and the modified approximants of the original sequence. This suggests the general question of *in which other situations do similar patterns of relationships exist?* In section 2, it was shown that (at least some of) this behavour extends to the setting of products of invertible elements in Banach algebras. From of [3] it is clear that there are some similar results available in the setting of topological groups. But more generally, are there other classes of sequences that diverge by oscillation, but for which "nice" asymptotics for the sequences exist thus enabling the computation of the sequential closure?

Even more generally, when "nice" asymptotics do not exist, is the sequential closure non-trivial and interesting or useful?

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