A q-CONTINUED FRACTION

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Abstract. Let a, b, c, d be complex numbers with $d \neq 0$ and $|q| < 1$. Define

$$H_1(a,b,c,d,q) := \frac{1}{1} + \frac{-abq+c}{(a+b)q+d} + \cdots + \frac{-abq^{2n+1}+cq^n}{(a+b)q^{n+1}+d} + \cdots.$$

We show that $H_1(a,b,c,d,q)$ converges and

$$\frac{1}{H_1(a,b,c,d,q)} - 1 = \frac{c-abq}{d+aq} \frac{\sum_{j=0}^{\infty} \frac{(b/d)^j (-c/bd)_j \, q^{j(j+3)/2}}{(q)_j(-aq^2/d)_j}}{\sum_{j=0}^{\infty} \frac{(b/d)^j (-c/bd)_j \, q^{j(j+1)/2}}{(q)_j(-aq/d)_j}}.$$

We then use this result to deduce various corollaries, including the following:

$$\frac{1}{1} - \frac{q}{1+q} - \frac{q^3}{1+q^2} - \frac{q^5}{1+q^3} - \cdots - \frac{q^{2n-1}}{1+q^n} - \cdots = \frac{(q^2;q^3)_\infty}{(q;q^3)_\infty},$$

$$(-aq)_\infty \sum_{j=0}^{\infty} \frac{(bq)^j(-c/b)_j q^{j(j-1)/2}}{(q)_j(-aq)_j} = (-bq)_\infty \sum_{j=0}^{\infty} \frac{(aq)^j(-c/a)_j q^{j(j-1)/2}}{(q)_j(-bq)_j},$$

and the Rogers-Ramanujan identities,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \qquad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

1. introduction

The work in the present paper was initially motivated by claims of Ramanujan about two related continued fractions.

The first claim concerns a curious continued fraction with three limits. To describe Ramanujan's claim, found his lost notebook ([19], p.45), we first need some notation. Throughout take $q \in \mathbb{C}$ with $|q| < 1$. The following

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standard notation for q-products will also be employed:

$$(a)_0 := (a;q)_0 := 1, \quad (a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-a\,q^k), \qquad \text{if } n \geq 1,$$

and

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-a\,q^k), \qquad\qquad |q|<1.$$

Set $\omega = e^{2\pi i/3}$. Ramanujan's claim was that, for $|q|<1$,
(1.1)

$$\lim_{n\to\infty}\left(\frac{1}{1}-\frac{1}{1+q}-\frac{1}{1+q^2}-\cdots-\frac{1}{1+q^n+a}\right) = -\omega^2\left(\frac{\Omega-\omega^{n+1}}{\Omega-\omega^{n-1}}\right)\cdot\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty},$$

where

$$\Omega := \frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q,q)_\infty}{(\omega q,q)_\infty}.$$

Ramanujan's notation is confusing, but what his claim means is that the limit exists as $n\to\infty$ in each of the three congruence classes modulo 3, and that the limit is given by the expression on the right side of (1.1). Ramanujan's claim is proved in a recent paper [7].

In [11] the first and second authors of the present paper generalized Ramanujan's result to produce q-continued fractions with m limits, where $m \geq 3$ is an arbitrary integer. Let ω be a primitive m-th root of unity and, for ease of notation, let $\bar{\omega} = 1/\omega$. Define

$$G(q) := \frac{1}{1}-\frac{1}{\omega+\bar{\omega}+q}-\frac{1}{\omega+\bar{\omega}+q^2}-\frac{1}{\omega+\bar{\omega}+q^3}+\cdots. \tag{1.2}$$

For $|q|<1$ and a, $x \neq 0$, define

$$P(a,x,q) := \sum_{j=0}^{\infty}\frac{q^{j(j+1)/2}a^j x^j}{(q;q)_j(x^2q;q)_j}. \tag{1.3}$$

In [11] the following theorem is proved.

Theorem 1. *Let ω be a primitive m-th root of unity and let $\bar{\omega}=1/\omega$. Let $1 \leq i \leq m$. Then*

$$\lim_{k\to\infty}\frac{1}{\omega+\bar{\omega}+q}-\frac{1}{\omega+\bar{\omega}+q^2}-\cdots\frac{1}{\omega+\bar{\omega}+q^{mk+i}} = \frac{\omega^{1-i}P(q,\omega,q)-\omega^{i-1}P(q,w^{-1},q)}{\omega^{-i}P(1,\omega,q)-\omega^{i}P(1,w^{-1},q)}. \tag{1.4}$$

The result is stated for the first tail of $G(q)$, rather than $G(q)$ itself, for aesthetic reasons. Ramanujan's continued fraction, with $a=0$, is the special case $m=6$ of (1.2). The result in the above theorem also appears in the paper of Ismail and Stanton [16].

If we make the transformation $q \to 1/q$ in Ramanujan's continued fraction

$$T(q) := \frac{1}{1} - \frac{1}{1+q} - \frac{1}{1+q^2} - \cdots - \frac{1}{1+q^n} - \cdots \tag{1.5}$$

and clear denominators, we get the continued fraction

$$S(q) := \frac{1}{1} - \frac{q}{1+q} - \frac{q^3}{1+q^2} - \frac{q^5}{1+q^3} - \cdots - \frac{q^{2n-1}}{1+q^n} - \cdots.$$

Remarkably, Ramanujan made a deep claim about $S(q)$ also, namely, that

$$S(q) = \frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}. \tag{1.6}$$

This claim is proved in [8], by the same group of authors who proved (1.1) in [7]. In this paper the authors remark that "Of the many continued fractions found by Ramanujan, (1.6) is, by far, the most difficult to prove." The only other proof, to date, can be found [6]. Amongst other things in this present paper, we also give a new proof of (1.6).

While the first and second authors of this current paper were working on generalizations of (1.5), computer investigations indicated to the second and third author that the numerator and denominator of $S(q)$ converged separately to $1/(q;q^3)_\infty$ and $1/(q^2;q^3)_\infty$, respectively. That this is indeed the case is not obvious in the proofs in [6] or [8]. While trying to prove these separate convergence results, at some point "the penny dropped" and we realized that we could adapt the method being used to generalize (1.5) by making the substitution $q \to 1/q$ at the right point and obtain the limit (as quotients of certain q-series) of a quite general class of q-continued fractions, a class which includes $S(q)$ above. Ramanujan's result (1.6) then followed after applying a special case of a powerful basic hypergeometric series transformation due to Watson.

On the way to our main result, we find the limit of continued fractions of the form

$$H(a,b,c,d,q) = \frac{1}{1} + \frac{-ab+cq}{a+b+dq} + \frac{-ab+cq^2}{a+b+dq^2} + \cdots + \frac{-ab+cq^n}{a+b+dq^n} + \cdots.$$

Several of the q-continued fractions in the literature arise as special cases of this continued fraction. However, we do not investigate this here as a closely related continued fraction, namely,

$$1+a+d+\frac{-a+cq}{a+1+dq} + \frac{-a+cq^2}{a+1+dq^2} + \cdots + \frac{-a+cq^n}{a+1+dq^n} + \cdots,$$

was investigated by Hirschhorn in [13] and [14], and many of the well-known continued fraction identities were derived by him as corollaries of his main result.

Our result for the continued fraction $H(a,b,c,d,q)$ could be derived from Hirschhorn's through various substitutions, equivalence transformations and series manipulations. However, it is perhaps just as simple to derive it directly, using the recurrence relations for the numerators and denominators and generating functions. More importantly, the derivation of our main result relies on finding expressions for the N-th numerator and denominator of $H(a,b,c,d,q)$ and then applying the transformation $q \to 1/q$.

Our main result is as follows (see Theorem 3 for a more complete statement): Let a, b, c, d be complex numbers with $d \neq 0$ and $|q| < 1$. Define

$$H_1(a,b,c,d,q) := \frac{1}{1} + \frac{-abq+c}{(a+b)q+d} + \cdots + \frac{-abq^{2n+1}+cq^n}{(a+b)q^{n+1}+d} + \cdots.$$

Then $H_1(a,b,c,d,q)$ converges and

$$\frac{1}{H_1(a,b,c,d,q)} - 1 = \frac{c-abq}{d+aq} \frac{\sum_{j=0}^{\infty} \dfrac{(b/d)^j(-c/bd)_j\, q^{j(j+3)/2}}{(q)_j(-aq^2/d)_j}}{\sum_{j=0}^{\infty} \dfrac{(b/d)^j(-c/bd)_j\, q^{j(j+1)/2}}{(q)_j(-aq/d)_j}}.$$

We then use this result, in combination with other well know transformations like the Jacobi triple product identity and the aforementioned transformation of Watson, to deduce various corollaries, some of which are originally due to Ramanujan.

$$\frac{1}{1} - \frac{q}{1+q} - \frac{q^3}{1+q^2} - \frac{q^5}{1+q^3} - \cdots - \frac{q^{2n-1}}{1+q^n} - \cdots = \frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}. \tag{1.7}$$

$$1 + \frac{aq}{1} + \frac{bq+e}{1} + \frac{aq^2}{1} + \frac{bq^2+e}{1} + \frac{aq^3}{1} + \frac{bq^3+e}{1} + \cdots = \frac{\displaystyle\sum_{j=0}^{\infty} \frac{(a/(e+1))^j\,(eq/(e+1))_j\; q^{j(j+1)/2}}{(q)_j\,(-bq/(e+1))_j}}{\displaystyle\sum_{j=0}^{\infty} \frac{(aq/(e+1))^j\,(e/(e+1))_j\; q^{j(j+1)/2}}{(q)_j\,(-bq/(e+1))_j}}. \tag{1.8}$$

$$1 + \frac{aq+e}{1} + \frac{bq}{1} + \frac{aq^2+e}{1} + \frac{bq^2}{1} + \frac{aq^3+e}{1} + \frac{bq^3}{1} + \cdots = \frac{(e+1)\displaystyle\sum_{j=0}^{\infty} \frac{(a/(e+1))^j\,(eb/(a(e+1)))_j\; q^{j(j+1)/2}}{(q)_j\,(-bq/(e+1))_j}}{\displaystyle\sum_{j=0}^{\infty} \frac{(aq/(e+1))^j\,(eb/(a(e+1)))_j\; q^{j(j+1)/2}}{(q)_j\,(-bq/(e+1))_j}}. \tag{1.9}$$

Remark: each of the previous two continued fractions generalizes a continued fraction of Ramanujan (see (4.2)).

(1.10)
$$(-aq)_\infty \sum_{j=0}^{\infty} \frac{(bq)^j(-c/b)_j\, q^{j(j-1)/2}}{(q)_j(-aq)_j} = (-bq)_\infty \sum_{j=0}^{\infty} \frac{(aq)^j(-c/a)_j\, q^{j(j-1)/2}}{(q)_j(-bq)_j}.$$

$$\frac{1}{1+}\,\frac{q+q^2}{1}\,\frac{}{+}\,\frac{q^2+q^4}{1}\,\frac{}{+}\,\frac{q^3+q^6}{1}\,\frac{}{+\cdots} = \frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3}. \tag{1.11}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}, \tag{1.12}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.$$

Remarks: (1) We could, without loss of generality in $H_1(a,b,c,d,q)$, set one of the parameters a, b, c or d equal to one and recover the general case, if desired, by a change of variables. However, it is better for our present purposes to retain the flexibility of having 4 parameters and not having deal with these changes of variables (see Corollaries 1 and 5, for example, where having this full flexibility allowed us to derive our results more easily).

(2) Shortly after we had proved our main result for $H_1(a,b,c,d,q)$ and obtained the various corollaries, the second author was browsing an early draft version of [5], which one of the authors of [5] had given him. The purpose was to find further applications of our main result about $H_1(a,b,c,d,q)$ and possibly give new proofs, or possibly generalizations (see Corollary 5), of some of Ramanujan's results. Instead he was surprised to find that Ramanujan had stated a result that was quite close to our main result for $H_1(a,b,c,d,q)$.

For any complex numbers a, b, λ and q, with $|q|<1$, define

$$G(a,\lambda;b;q) = \sum_{n=0}^{\infty}\frac{(-\lambda/a;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}.$$

Entry 6.4.4 (p.43) We have

$$\frac{G(aq,\lambda q;b;q)}{G(a,\lambda;b;q)} = \frac{1}{1+aq}\,\frac{}{+}\,\frac{\lambda q - abq^2}{1+q(aq+b)}\,\frac{}{+}\,\frac{\lambda q^2 - abq^4}{1+q^2(aq+b)} \\ \frac{}{+\cdots+}\,\frac{}{+}\,\frac{\lambda q^n - abq^{2n}}{1+q^n(aq+b)}\,\frac{}{+\cdots}. \tag{1.13}$$

It seems clear that our main result concerning $H_1(a,b,c,d,q)$ could also be derived from (1.13), after various changes of variable and q-series manipulations, but possibly proving it the way we did may be more illuminating.

2. Proofs

We recall the q-binomial theorem [3], pp. 35–36.

Lemma 1. *If* $\begin{bmatrix} n \\ m \end{bmatrix}$ *denotes the Gaussian polynomial defined by*

$$\begin{bmatrix} n \\ m \end{bmatrix} := \begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}, & \text{if } 0 \le m \le n, \\ 0, & \text{otherwise} , \end{cases}$$

then

$$(z;q)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^j z^j q^{j(j-1)/2}, \tag{2.1}$$

$$\frac{1}{(z;q)_N} = \sum_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix} z^j.$$

Note for later use that

$$\begin{bmatrix} n \\ m \end{bmatrix}_{1/q} = q^{m(m-n)} \begin{bmatrix} n \\ m \end{bmatrix}_q .$$

Theorem 2. *Let*

$$H(a,b,c,d,q) = \frac{1}{1} + \frac{-ab+cq}{a+b+dq} + \frac{-ab+cq^2}{a+b+dq^2} + \cdots + \frac{-ab+cq^n}{a+b+dq^n} + \cdots .$$

(i) Let $A_N := A_N(q)$ and $B_N := B_N(q)$ denote the N-th numerator convergent and N-th denominator convergent, respectively, of $H(a,b,c,d,q)$. Then A_N and B_N are given explicitly by the following formulae.

$$A_N = \tag{2.2}$$
$$\sum_{j,l,n \ge 0} a^j b^{N-1-n-j-l} c^l d^{n-l} q^{n(n+1)/2 + l(l+1)/2} \begin{bmatrix} n+j \\ j \end{bmatrix} \begin{bmatrix} N-1-j-l \\ n \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix}.$$

For $N \ge 2$,

$$B_N = A_N + (cq - ab) \times \tag{2.3}$$
$$\sum_{j,l,n \ge 0} a^j b^{N-2-n-j-l} c^l d^{n-l} q^{n(n+3)/2 + l(l+1)/2} \begin{bmatrix} n+j \\ j \end{bmatrix} \begin{bmatrix} N-2-j-l \\ n \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix}.$$

(ii) If $|a/b| < 1$ and $|q| < 1$ then $H(a,b,c,d,q)$ converges and

$$\frac{1}{H(a,b,c,d,q)} - 1 = \frac{(cq/b - a) \displaystyle\sum_{n=0}^{\infty} \frac{(d/b)^n q^{n(n+3)/2} (-cq/db)_n}{(a/b)_{n+1} (q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{(d/b)^n q^{n(n+1)/2} (-cq/db)_n}{(a/b)_{n+1} (q)_n}}. \tag{2.4}$$

(iii) If $|q|<1$, $|a|<1$ and $b=1$, then the numerators and denominators converge separately and

$$\lim_{N\to\infty} A_N = \sum_{n=0}^{\infty} \frac{d^n q^{n(n+1)/2}(-cq/d)_n}{(a)_{n+1}(q)_n}, \tag{2.5}$$

$$\lim_{N\to\infty} B_N = \sum_{n=0}^{\infty} \frac{d^n q^{n(n+1)/2}(-cq/d)_n}{(a)_{n+1}(q)_n} + (cq-a)\sum_{n=0}^{\infty} \frac{d^n q^{n(n+3)/2}(-cq/d)_n}{(a)_{n+1}(q)_n}. \tag{2.6}$$

Remarks: (a) By symmetry the conditions an a and b in (ii) and (iii) can be interchanged, in which case a and b are interchanged on the right sides. (b) The left side in (ii) is displayed as shown to simplify the representation on the right side.

Proof. (i) We follow the method of Hirschhorn in [13]. We suppose initially that $|q|<1$ and for ease of notation, let $A_N := A_N(q)$ and $B_N := B_N(q)$ and set $F(t)=\sum_{N\geq 1} A_N t^n$ and $G(t)=\sum_{N\geq 1} B_N t^n$. We suppose initially that a and b are chosen so that the series defining $F(t)$ and $G(t)$ converge ($|a|=|b|=1$ suffices for this). From the recurrence relations for the convergents of a continued fraction, we have that

$$A_{N+1} = (a+b+dq^N)A_N + (-ab+cq^N)A_{N-1}$$

holds for $N\geq 1$. If this equation is multiplied by t^{N+1} and summed over $N\geq 1$, we get

$$F(t)-t = t(a+b)F(t) + tdF(tq) - abt^2F(t) + ct^2qF(tq).$$

Here we have used $A_1=1$ and $A_0=0$. The last equation can be rewritten to give

$$F(t) = \frac{t}{(1-at)(1-bt)} + \frac{t(d+ctq)}{(1-at)(1-bt)}F(tq).$$

Next, this equation is iterated and we use the fact that $F(0)=0$ to get that

$$F(t) = \sum_{n\geq 1} \frac{t^n d^{n-1}(-ctq/d)_{n-1} q^{n(n-1)/2}}{(at)_n(bt)_n}.$$

The q-binomial theorem (Lemma 1) is applied to the q-products in the expression above to give that

$$F(t) = \sum_{\substack{n\geq 1,\\ j,k,l\geq 0}} t^{n+j+k+l} d^{n-1-l} a^j b^k c^l q^{\frac{n(n-1)}{2}+\frac{l(l+1)}{2}} \begin{bmatrix} n+j-1 \\ j \end{bmatrix} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \begin{bmatrix} n-1 \\ l \end{bmatrix}$$

$$= \sum_{j,k,l,n\geq 0} t^{n+1+j+k+l} d^{n-l} a^j b^k c^l q^{\frac{n(n+1)}{2}+\frac{l(l+1)}{2}} \begin{bmatrix} n+j \\ j \end{bmatrix} \begin{bmatrix} n+k \\ n \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix}.$$

Finally, we let $N = n + j + k + l + 1$, substitute for k and use the definition of $F(t)$ to get (2.2).

By similar reasoning we get that

$$\begin{aligned}
G(t) &= \sum_{n\geq 1} \frac{t^n d^{n-1}(-ctq/d)_{n-1} q^{n(n-1)/2}}{(at)_n (bt)_n} (1 + (cq - ab)tq^{n-1}) \\
&= F(t) + (cq - ab) \sum_{n\geq 1} \frac{t^{n+1} d^{n-1}(-ctq/d)_{n-1} q^{(n-1)(n+2)/2}}{(at)_n (bt)_n} \\
&= F(t) + (cq - ab) \sum_{n\geq 0} \frac{t^{n+2} d^{n}(-ctq/d)_{n} q^{n(n+3)/2}}{(at)_{n+1} (bt)_{n+1}} \\
&= F(t) + (cq - ab) \sum_{j,k,l,n\geq 0} t^{n+2+j+k+l} d^{n-l} a^j b^k c^l q^{\frac{n(n+3)}{2} + \frac{l(l+1)}{2}} \\
&\qquad \times \begin{bmatrix} n+j \\ j \end{bmatrix} \begin{bmatrix} n+k \\ n \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix}.
\end{aligned}$$

We let $N = n + j + k + l + 2$, substitute for k and use the definition of $G(t)$ to get (2.3).

(ii) The expression for A_N in (2.2) can be re-written as

(2.7)
$$\begin{aligned}
A_N = b^{N-1} \sum_{n\geq 0} (d/b)^n q^{n(n+1)/2} \sum_{j\geq 0} \begin{bmatrix} n+j \\ j \end{bmatrix} (a/b)^j \\
\times \sum_{l\geq 0} \begin{bmatrix} N-1-j-l \\ n \end{bmatrix} q^{l(l-1)/2} \left(\frac{cq}{bd}\right)^l \begin{bmatrix} n \\ l \end{bmatrix}.
\end{aligned}$$

By the definition of the Gaussian polynomials in Lemma 1, j, l and n are restricted by $l \leq n$ and $l + j + n \leq N - 1$.

Similarly, the expression for B_N in (2.3) can be re-written as

$$\begin{aligned}
B_N = A_N + b^{N-1}(cq/b - a) \sum_{n\geq 0} (d/b)^n q^{n(n+3)/2} \sum_{j\geq 0} \begin{bmatrix} n+j \\ j \end{bmatrix} (a/b)^j \\
\times \sum_{l\geq 0} \begin{bmatrix} N-2-j-l \\ n \end{bmatrix} q^{l(l-1)/2} \left(\frac{cq}{bd}\right)^l \begin{bmatrix} n \\ l \end{bmatrix}.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{N\to\infty} \frac{B_N - A_N}{b^{N-1}} = (cq/b - a) \sum_{n\geq 0} (d/b)^n q^{n(n+3)/2} \sum_{j\geq 0} \begin{bmatrix} n+j \\ j \end{bmatrix} (a/b)^j \\
\times \sum_{l=0}^{n} \frac{q^{l(l-1)/2} \left(\frac{cq}{bd}\right)^l}{(q)_n} \begin{bmatrix} n \\ l \end{bmatrix}
\end{aligned}$$

$$= (cq/b - a) \sum_{n=0}^{\infty} \frac{(d/b)^n q^{n(n+3)/2} (-cq/db)_n}{(a/b)_{n+1} (q)_n}.$$

Likewise the expression for A_N above gives that

$$\lim_{N\to\infty} \frac{A_N}{b^{N-1}} = \sum_{n=0}^{\infty} \frac{(d/b)^n q^{n(n+1)/2} (-cq/db)_n}{(a/b)_{n+1} (q)_n}.$$

Equation 2.4 is now immediate.

(iii) If we now set $b = 1$ the limit above, we get immediately that

$$\lim_{N\to\infty} A_N = \sum_{n\geq 0} \frac{d^n q^{n(n+1)/2} (-cq/d)_n}{(a)_{n+1} (q)_n}.$$

This proves (2.5). The proof of (2.6) is similar. □

We now prove our main theorem.

Theorem 3. *Let a, b, c, d be complex numbers with $d \neq 0$ and $|q| < 1$. Define*

$$H_1(a,b,c,d,q) := \frac{1}{1} + \frac{-abq + c}{(a+b)q + d} + \cdots + \frac{-abq^{2n+1} + cq^n}{(a+b)q^{n+1} + d} + \cdots.$$

(i) Let $C_N := C_N(q)$ and $D_N := D_N(q)$ denote the N-th numerator convergent and N-th denominator convergent, respectively, of $H_1(a,b,c,d,q)$. Then C_N and D_N are given explicitly by the formulae

$$C_N = d^{N-1} \sum_{j,\,l,\,n\geq 0} a^j b^{n-j-l} c^l d^{-n-l} q^{n(n+1)/2 + l(l-1)/2} \times \begin{bmatrix} N-1-n+j \\ j \end{bmatrix}_q \begin{bmatrix} N-1-j-l \\ n-j-l \end{bmatrix}_q \begin{bmatrix} N-1-n \\ l \end{bmatrix}_q. \tag{2.8}$$

For $N \geq 2$,

$$D_N = C_N + (c/bq - a) \sum_{j,\,l,\,n\geq 0} a^j b^{n+1-j-l} c^l d^{N-2-n-l} \times q^{(n+1)(n+2)/2 + l(l-1)/2} \begin{bmatrix} N-2-n+j \\ j \end{bmatrix}_q \begin{bmatrix} N-2-j-l \\ n-j-l \end{bmatrix}_q \begin{bmatrix} N-2-n \\ l \end{bmatrix}_q. \tag{2.9}$$

(ii) If $|q| < 1$ then $H_1(a,b,c,d,q)$ converges and

$$\frac{1}{H_1(a,b,c,d,q)} - 1 = \frac{c - abq}{(d+aq)q} \frac{\sum_{j=0}^{\infty} \dfrac{(b/d)^j (-c/bd)_j \, q^{(j+1)(j+2)/2}}{(q)_j (-aq^2/d)_j}}{\sum_{j=0}^{\infty} \dfrac{(b/d)^j (-c/bd)_j \, q^{j(j+1)/2}}{(q)_j (-aq/d)_j}}. \tag{2.10}$$

(iii) If $|q| < 1$ and $d = 1$, then the numerators and denominators converge separately and

$$C_\infty := \lim_{N\to\infty} C_N = (-aq)_\infty \sum_{j=0}^{\infty} \frac{q^{j(j+1)/2} b^j (-c/b)_j}{(q)_j(-aq)_j}. \tag{2.11}$$

(2.12)

$$D_\infty := \lim_{N\to\infty} D_N = C_\infty + (c/q - ab)(-aq)_\infty \sum_{j=0}^{\infty} \frac{q^{(j+1)(j+2)/2} b^j (-c/b)_j}{(q)_j(-aq)_{j+1}}.$$

Proof. The continued fraction $H_1(a,b,c,d,q)$ is derived from $H(a,b,c,d,q)$ by making the substitution $q \to 1/q$ and then applying a sequence of similarity transformations to clear the negative powers of q. Thus,

$$\begin{aligned}
C_N &= q^{N(N-1)/2} A_N(1/q),\\
&= \sum_{j,l,n\geq 0} a^j b^{N-1-n-j-l} c^l d^{n-l} q^{(N-n)(N-n-1)/2+l(l-1)/2}\\
&\qquad\qquad \times \begin{bmatrix} n+j \\ j \end{bmatrix}_q \begin{bmatrix} N-1-j-l \\ n \end{bmatrix}_q \begin{bmatrix} n \\ l \end{bmatrix}_q\\
&= \sum_{j,l,n\geq 0} a^j b^{n-j-l} c^l d^{N-1-n-l} q^{n(n+1)/2+l(l-1)/2}\\
&\qquad\qquad \times \begin{bmatrix} N-1-n+j \\ j \end{bmatrix}_q \begin{bmatrix} N-1-j-l \\ n-j-l \end{bmatrix}_q \begin{bmatrix} N-1-n \\ l \end{bmatrix}_q\\
&= d^{N-1} \sum_{n=0}^{N-1} \sum_{j=0}^{n} \sum_{l=0}^{\min(n-j,\, N-1-n)} a^j b^{n-j-l} c^l d^{-n-l} q^{n(n+1)/2+l(l-1)/2}\\
&\qquad\qquad \times \begin{bmatrix} N-1-n+j \\ j \end{bmatrix}_q \begin{bmatrix} N-1-j-l \\ n-j-l \end{bmatrix}_q \begin{bmatrix} N-1-n \\ l \end{bmatrix}_q.
\end{aligned}$$

For the next-to-last step we replaced n by $N-1-n$ and in the last step the upper limits on j, l and n come from the definition of the Gaussian polynomials in Lemma 1. This proves (2.8).

Similarly,

$$\begin{aligned}
D_N &= q^{N(N-1)/2} B_N(1/q)\\
&= C_N + (c/bq - a) \sum_{j,l,n\geq 0} a^j b^{N-1-n-j-l} c^l d^{n-l} q^{(N-n)(N-n-1)/2+l(l-1)/2}\\
&\qquad\qquad \times \begin{bmatrix} n+j \\ j \end{bmatrix}_q \begin{bmatrix} N-2-j-l \\ n \end{bmatrix}_q \begin{bmatrix} n \\ l \end{bmatrix}_q\\
&= C_N + (c/bq - a) \sum_{j,l,n\geq 0} a^j b^{n+1-j-l} c^l d^{N-2-n-l} q^{(n+1)(n+2)/2+l(l-1)/2}\\
&\qquad\qquad \times \begin{bmatrix} N-2-n+j \\ j \end{bmatrix}_q \begin{bmatrix} N-2-j-l \\ n-j-l \end{bmatrix}_q \begin{bmatrix} N-2-n \\ l \end{bmatrix}_q
\end{aligned}$$

$$= C_N + (c/bq - a) \sum_{n=0}^{N-2} \sum_{j=0}^{n} \sum_{l=0}^{\min(n-j,\ N-2-n)} a^j b^{n+1-j-l} c^l d^{N-2-n-l} \times$$
$$q^{(n+1)(n+2)/2+l(l-1)/2} \begin{bmatrix} N-2-n+j \\ j \end{bmatrix}_q \begin{bmatrix} N-2-j-l \\ n-j-l \end{bmatrix}_q \begin{bmatrix} N-2-n \\ l \end{bmatrix}_q .$$

The second equality follows upon replacing n by $N-n-2$ and the bounds on j, l and n in the last equality follow, as above, from the definition of the Gaussian polynomials in Lemma 1. This proves (2.9).

From (2.9),

$$\lim_{N\to\infty} \frac{D_N - C_N}{d^{N-1}} = \frac{c/q - ab}{d} \sum_{j,\,l,\,n\geq 0} a^j b^{n-j-l} c^l d^{-n-l} \frac{q^{(n+1)(n+2)/2+l(l-1)/2}}{(q)_j (q)_{n-j-l} (q)_l}$$
$$= \frac{c/q - ab}{d} \sum_{n\geq 0} (b/d)^n q^{(n+1)(n+2)/2} \sum_{j=0}^{n} \frac{(a/b)^j}{(q)_j (q)_{n-j}}$$
$$\times \sum_{l=0}^{n-j} q^{l(l-1)/2} (c/bd)^l \begin{bmatrix} n-j \\ l \end{bmatrix}$$
$$= \frac{c - abq}{dq} \sum_{n\geq 0} (b/d)^n q^{(n+1)(n+2)/2} \sum_{j=0}^{n} \frac{(a/b)^j (-c/bd)_{n-j}}{(q)_j (q)_{n-j}}$$
$$= \frac{c - abq}{dq} \sum_{n\geq 0} (b/d)^n q^{(n+1)(n+2)/2} \sum_{j=0}^{n} \frac{(a/b)^{n-j} (-c/bd)_j}{(q)_j (q)_{n-j}}$$
$$= \frac{c - abq}{dq} \sum_{j=0}^{\infty} \frac{(b/a)^j (-c/bd)_j}{(q)_j} \sum_{n\geq j} \frac{(a/d)^n q^{(n+1)(n+2)/2}}{(q)_{n-j}}$$
$$= \frac{c - abq}{dq} \sum_{j=0}^{\infty} \frac{(b/a)^j (-c/bd)_j}{(q)_j} \sum_{n\geq 0} \frac{(a/d)^{n+j} q^{n(n+3)/2+jn+(j+1)(j+2)/2}}{(q)_n}$$
$$= \frac{c - abq}{dq} \sum_{j=0}^{\infty} \frac{(b/d)^j (-c/bd)_j q^{(j+1)(j+2)/2}}{(q)_j} \sum_{n\geq 0} \frac{(aq^{j+2}/d)^n q^{n(n-1)/2}}{(q)_n}$$
$$= \frac{c - abq}{dq} \sum_{j=0}^{\infty} \frac{(b/d)^j (-c/bd)_j q^{(j+1)(j+2)/2}}{(q)_j} (-aq^{j+2}/d)_\infty$$
$$= \frac{c - abq}{dq} (-aq^2/d)_\infty \sum_{j=0}^{\infty} \frac{(b/d)^j (-c/bd)_j \, q^{(j+1)(j+2)/2}}{(q)_j (-aq^2/d)_j}.$$

It follows similarly from (2.8) that

$$\lim_{N\to\infty} \frac{C_N}{d^{N-1}} = \sum_{j,\,l,\,n\geq 0} a^j b^{n-j-l} c^l d^{-n-l} \frac{q^{n(n+1)/2+l(l-1)/2}}{(q)_j (q)_{n-j-l} (q)_l}$$

$$= (-aq/d)_\infty \sum_{j=0}^{\infty} \frac{(b/d)^j (-c/bd)_j \, q^{j(j+1)/2}}{(q)_j (-aq/d)_j}.$$

We omit the details. That (2.10) holds is now immediate.

That (2.11) and (2.12) hold follows immediately upon letting $d = 1$ in the limits above. □

3. New Proofs of Some Continued fraction Identities

In what follows, we make some use of the Jacobi Triple Product Identity.

Theorem 4. *For $|q| < 1$ and $z \in \mathbb{C}\backslash\{0\}$,*

$$(-qz; q^2)_\infty (-q/z; q^2)_\infty (q^2; q^2)_\infty = \sum_{n=-\infty}^{\infty} (-z)^n q^{n^2}.$$

We also need a result of Watson on basic hypergeometric series. An ${}_r\phi_s$ basic hypergeometric series is defined by

$${}_r\phi_s \left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \ldots (a_r;q)_n}{(q;q)_n (b_1;q)_n \ldots (b_s;q)_n} \left((-1)^n q^{n(n-1)/2} \right)^{s+1-r} x^n,$$

for $|q| < 1$.

Watson's theorem is that

$${}_8\phi_7 \left(\begin{matrix} A,\ q\sqrt{A},\ -q\sqrt{A},\ B,\ C,\ D,\ E,\ q^{-n} \\ \sqrt{A},\ -\sqrt{A},\ Aq/B,\ Aq/C,\ Aq/D,\ Aq/E,\ Aq^{n+1} \end{matrix} ; q, \frac{A^2 q^{n+2}}{BCDE} \right) = \frac{(Aq)_n (Aq/DE)_n}{(Aq/D)_n (Aq/E)_n} \, {}_4\phi_3 \left(\begin{matrix} Aq/BC, D, E, q^{-n} \\ Aq/B, Aq/C, DEq^{-n}/A \end{matrix} ; q, q \right),$$

where n is a non-negative integer. If we let B, D and $n \to \infty$ (as in [13]), we get

$$\sum_{r \geq 0} \frac{(1 - Aq^{2r})(A)_r (C)_r (E)_r \left(-A^2/CE\right)^r q^{3r(r-1)/2+2r}}{(1-A)(Aq/C)_r (Aq/E)_r (q)_r} = \frac{(Aq)_\infty}{(Aq/E)_\infty} \sum_{r \geq 0} \frac{(E)_r (-Aq/E)^r q^{r(r-1)/2}}{(q)_r (Aq/C)_r}.$$

If we set $A = c/d^2$, $C = -c/ad$ and $E = -c/bd$, we get that

(3.1)
$$\lim_{N \to \infty} \frac{C_N}{d^{N-1}} = (-aq/d)_\infty \sum_{j=0}^{\infty} \frac{(bq/d)^j (-c/bd)_j \, q^{j(j-1)/2}}{(q)_j (-aq/d)_j} = \frac{(-aq/d)_\infty (-bq/d)_\infty}{(cq/d^2)_\infty}$$

$$\times \sum_{r\geq 0} \frac{(1-cq^{2r}/d^2)(-c/ad)_r(-c/bd)_r(c/d^2)_r\left(-ab/d^2\right)^r q^{3r(r-1)/2+2r}}{(1-c/d^2)(-aq/d)_r(-bq/d)_r(q)_r}.$$

Likewise, if we set $A = cq/d^2$, $C = -c/ad$ and $E = -c/bd$, we get that

(3.2)

$$\lim_{N\to\infty} \frac{D_N - C_N}{d^{N-1}} = \frac{c-abq}{d}(-aq^2/d)_\infty \sum_{j=0}^{\infty} \frac{(bq^2/d)^j(-c/bd)_j\, q^{j(j-1)/2}}{(q)_j(-aq^2/d)_j}$$

$$= \frac{c-abq}{d}\frac{(-aq^2/d)_\infty(-bq^2/d)_\infty}{(cq^2/d^2)_\infty}$$

$$\times \sum_{r\geq 0} \frac{(1-cq^{2r+1}/d^2)(-c/ad)_r(-c/bd)_r(cq/d^2)_r\left(-abq^2/d^2\right)^r q^{3r(r-1)/2+2r}}{(1-cq/d^2)(-aq^2/d)_r(-bq^2/d)_r(q)_r}.$$

We note the first equalities in (3.1) and (3.2) imply that the somewhat amusing identity

$$\frac{1}{1}+\frac{abq+bd}{(a-b)q+d}+\cdots+\frac{abq^{2n+1}+bdq^n}{(a-b)q^{n+1}+d}+\cdots = \frac{1}{1+b}$$

holds for all complex numbers a, b and d and all q with $|q| < 1$ and $d \neq 0, -aq^n$, $n \geq 1$. This follows upon setting $c = -bd$ and then replacing b by $-b$. This result also follows from the following theorem of Pincherle [18] :

Theorem 5. *(Pincherle) Let* $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *and* $\{G_n\}_{n=-1}^{\infty}$ *be sequences of real or complex numbers satisfying* $a_n \neq 0$ *for* $n \geq 1$ *and for all* $n \geq 1$,

$$G_n = a_n G_{n-2} + b_n G_{n-1}. \tag{3.3}$$

Let $\{B_n\}_{n=1}^{\infty}$ *denote the denominator convergents of the continued fraction* $\operatorname*{K}_{n=1}^{\infty} \frac{a_n}{b_n}$.

If $\lim_{n\to\infty} G_n/B_n = 0$ *then* $\operatorname*{K}_{n=1}^{\infty} \frac{a_n}{b_n}$ *converges and its limit is* $-G_0/G_{-1}$.

However, Pincherle's theorem is less informative in that it does not show that the numerators converge to $(-aq;q)_\infty$ and that the denominators converge to $(1+b)(-aq;q)_\infty$.

The symmetry in a and b in the continued fraction of Theorem 3 can be exploited to prove something a little less trivial.

Corollary 1. *Let* a, b, c *and* q *be complex numbers with* $|q| < 1$ *and* $a \neq 0$. *Then*

(3.4)

$$(-aq)_\infty \sum_{j=0}^{\infty} \frac{(bq)^j(-c/b)_j\, q^{j(j-1)/2}}{(q)_j(-aq)_j} = (-bq)_\infty \sum_{j=0}^{\infty} \frac{(aq)^j(-c/a)_j\, q^{j(j-1)/2}}{(q)_j(-bq)_j}.$$

If, in addition, $1 - bq^n \neq 0$ *for* $n \geq 1$*, then*

$$\sum_{j=0}^{\infty} \frac{(-b/a;q)_j a^j \, q^{j(j+1)/2}}{(q)_j (bq)_j} = \frac{(-aq;q)_\infty}{(bq;q)_\infty}. \tag{3.5}$$

The identity at (3.4) is found in Ramanujan's lost notebook [19] and a proof can be found in the recent book by Andrews and Berndt [5].

The second identity is found in Ramanujan's notebooks (see [9], Chapter 27, Entry 1, page 262). This identity is also equivalent to a result found in Andrews [2], where Andrews attributes it to Cauchy.

Proof. Let $d = 1$ in Theorem 3. The symmetry in a and b in the continued fraction in Theorem 3 and the first equality in (3.1) give (3.4) immediately.

Ramanujan's result (3.5) follows after setting $c = -b$ and then replacing b by $-b$. □

We now prove some continued fraction identities.

Corollary 2. *If* $|q| < 1$*, then*

$$\frac{1}{1} - \frac{q}{q+1} - \frac{q^3}{q^2+1} - \cdots - \frac{q^{2n-1}}{q^n+1} - \cdots = \frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}, \tag{3.6}$$

with the numerators converging to $1/(q;q^3)_\infty$ *and the denominators converging to* $1/(q^2;q^3)_\infty$*.*

This continued fraction is due to Ramanujan and can be found in his second notebook ([20], page 290). It has been proved in [6] and in [8]. Both proofs are quite difficult and it is not shown that the numerators and denominators converge separately. We feel this present proof is simpler, although it uses the power of Watson's Theorem.

Proof. Let $\omega = \exp(2\pi\imath/3)$ and set $a = -\omega$, $b = -\omega^2$, $c = 0$ and $d = 1$ in Theorem 3, so that the continued fraction in the theorem is the continued fraction in Corollary 2. By the second equality in (3.1),

$$\begin{aligned}
\lim_{n\to\infty} C_n &= (\omega q)_\infty (\omega^2 q)_\infty \sum_{r\geq 0} \frac{\left(-q^2\right)^r q^{3r(r-1)/2}}{(\omega q)_r (\omega^2 q)_r (q)_r} \\
&= (\omega q)_\infty (\omega^2 q)_\infty \sum_{r\geq 0} \frac{\left(-q^2\right)^r (q^3)^{r(r-1)/2}}{(q^3;q^3)_r} \\
&= (\omega q)_\infty (\omega^2 q)_\infty (q^2;q^3)_\infty \\
&= \frac{(\omega q)_\infty (\omega^2 q)_\infty (q)_\infty (q^2;q^3)_\infty}{(q)_\infty} \\
&= \frac{(q^3;q^3)_\infty (q^2;q^3)_\infty}{(q)_\infty} \\
&= \frac{1}{(q;q^3)_\infty}
\end{aligned}$$

The third equality above follows from the q-binomial theorem. By the second equality in (3.2)

$$\begin{aligned}
\lim_{n\to\infty} D_n - C_n &= -q(\omega q^2)_\infty(\omega^2 q^2)_\infty \sum_{r\geq 0} \frac{(-1)^r\, q^{3r(r-1)/2+4r}}{(\omega q^2)_r(\omega^2 q^2)_r(q)_r} \\
&= -q(\omega q)_\infty(\omega^2 q)_\infty \sum_{r\geq 0} \frac{(-1)^r\, q^{3r(r-1)/2+4r}(1-q^{r+1})}{(\omega q)_{r+1}(\omega^2 q)_{r+1}(q)_{r+1}} \\
&= q(\omega q)_\infty(\omega^2 q)_\infty \sum_{r\geq 1} \frac{(-1)^r\, q^{(3r^2-r)/2-1}(1-q^r)}{(\omega q)_r(\omega^2 q)_r(q)_r} \\
&= (\omega q)_\infty(\omega^2 q)_\infty \sum_{r\geq 1} \frac{(-1)^r\, q^{(3r^2-3r)/2}(q^r-q^{2r})}{(\omega q)_r(\omega^2 q)_r(q)_r} \\
&= (\omega q)_\infty(\omega^2 q)_\infty \sum_{r\geq 0} \frac{(-1)^r\, (q^3)^{r(r-1)/2}(q^r-q^{2r})}{(q^3;q^3)_r} \\
&= (\omega q)_\infty(\omega^2 q)_\infty((q;q^3)_\infty - (q^2;q^3)_\infty)
\end{aligned}$$

Here again we have used the q-binomial theorem. From the third expression above for $\lim_{n\to\infty} C_n$, it follows that

$$\begin{aligned}
\lim_{n\to\infty} D_n &= (\omega q)_\infty(\omega^2 q)_\infty(q;q^3)_\infty \\
&= \frac{(\omega q)_\infty(\omega^2 q)_\infty(q)_\infty(q;q^3)_\infty}{(q)_\infty} \\
&= \frac{(q^3;q^3)_\infty(q;q^3)_\infty}{(q)_\infty} \\
&= \frac{1}{(q^2;q^3)_\infty}
\end{aligned}$$

□

Corollary 3. *If* $|q|<1$*, then*

$$S(q) := \frac{1}{1+}\,\frac{q+q^2}{1}\,\frac{}{+}\,\frac{q^2+q^4}{1}\,\frac{}{+}\,\frac{q^3+q^6}{1}\,\frac{}{+\cdots} = \frac{(q;q^2)_\infty}{(q^3;q^6)^3_\infty}. \tag{3.7}$$

This continued fraction is also due to Ramanujan and can be found in his second notebook ([20], page 373). The first proofs in print are due to Watson [26] and Selberg [24]. Other proofs are due to Gordon [12], Andrews [1] and Hirschhorn [15].

Proof. Set $a=-1/q^{1/2}$, $b=1/q^{1/2}$, $c=1$ and $d=1$ in Theorem 3, so that the continued fraction in the theorem is

$$\frac{1}{1+2S(q)}.$$

By the second equality in (3.1),

$$\begin{aligned}
\lim_{n\to\infty} C_n &= \frac{(q^{1/2})_\infty(-q^{1/2})_\infty}{(q)_\infty} \\
&\quad\times \left(1+\sum_{r\geq 1}\frac{(1-q^{2r})(q^{1/2})_r(-q^{1/2})_r(q)_{r-1}q^{3r(r-1)/2+r}}{(q^{1/2})_r(-q^{1/2})_r(q)_r}\right) \\
&= \frac{(q;q^2)_\infty}{(q)_\infty}\left(1+\sum_{r\geq 1}(1+q^r)q^{(3r^2-r)/2}\right) \\
&= \frac{(q;q^2)_\infty}{(q)_\infty}\sum_{r=-\infty}^{\infty}(q^{3/2})^{r^2}(q^{1/2})^r \\
&= \frac{(q;q^2)_\infty}{(q)_\infty}(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty.
\end{aligned}$$

The last equality above follows from the Jacobi triple product identity. By the second equality in (3.2),

$$\begin{aligned}
\lim_{n\to\infty} D_n - C_n &= 2\frac{(q^{3/2})_\infty(-q^{3/2})_\infty}{(q^2)_\infty} \\
&\quad\times\sum_{r\geq 0}\frac{(1-q^{2r+1})(q^{1/2})_r(-q^{1/2})_r(q)_r q^{3r(r-1)/2+3r}}{(1-q)(q^{3/2})_r(-q^{3/2})_r(q)_r} \\
&= 2\frac{(q;q^2)_\infty}{(q)_\infty}\sum_{r\geq 0}q^{3r^2/2+3r/2} \\
&= \frac{(q;q^2)_\infty}{(q)_\infty}\sum_{r=-\infty}^{\infty}(q^{3/2})^{r^2}(q^{3/2})^r \\
&= \frac{(q;q^2)_\infty}{(q)_\infty}(-1;q^3)_\infty(-q^3;q^3)_\infty(q^3;q^3)_\infty \\
&= 2\frac{(q;q^2)_\infty}{(q)_\infty}(-q^3;q^3)_\infty(-q^3;q^3)_\infty(q^3;q^3)_\infty
\end{aligned}$$

Here again we have used the Jacobi triple product theorem.

The result now follows after a little algebra and some elementary infinite product manipulations. □

We next give a proof of the Rogers-Ramanujan identities. Hirschhorn has given a proof in [14] of these identities that is equivalent to setting $a = b = 0$, $c = 1$ and $d = 0$ in (3.1) and (3.2), whereupon the identities follow after applying the Jacobi Triple product identity. We include this proof because of its elegance. We also give a different proof for the infinite product representations for the numerator and denominator (see Corollary 4 below). For this we need two identities, one due to Rogers [22] and one due to Andrews [4] (identities A.44 and A.62 in Slater's list [25]):

$$\sum_{r=0}^{\infty} \frac{q^{3r(r+1)/2}}{(q;q^2)_{r+1}(q;q)_r} = \frac{(q^8;q^{10})_\infty (q^2;q^{10})_\infty (q^{10};q^{10})_\infty}{(q;q)_\infty} \tag{3.8}$$

$$\sum_{r=0}^{\infty} \frac{(-q;q)_r q^{r(3r+1)/2}}{(q;q)_{2r+1}} = \frac{(q^6;q^{10})_\infty (q^4;q^{10})_\infty (q^{10};q^{10})_\infty}{(q;q)_\infty} \tag{3.9}$$

Corollary 4. *Let $A_n(q)$ and $B_n(q)$ denote the n-th numerator and denominator, respectively of the continued fraction*

$$K(q) = 1 + \frac{q}{1} \, {}_{+} \, \frac{q^2}{1} \, {}_{+} \, \frac{q^3}{1} \, {}_{+} \, \cdots .$$

Then

$$\lim_{n\to\infty} A_n(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{3.10}$$

$$\lim_{n\to\infty} B_n(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

These identities were first proved by L.J. Rogers in 1894 [21] in a paper that was completely ignored. They were rediscovered (without proof) by Ramanujan sometime before 1913. In 1917, Ramanujan rediscovered Roger's paper while browsing a journal. Also in 1917, these identities were rediscovered and proved independently by Issai Schur [23]. There are now many different proofs.

Proof. First set $a = b = 0$, $c = 1$ and $d = 1$ in Theorem 3, so that the continued fraction in the theorem is

$$\frac{1}{1} \, {}_{+} \, \frac{1}{K(q)}.$$

By the first equalities at (3.1) and (3.2),

$$\lim_{n\to\infty} A_n(q) = \lim_{n\to\infty} C_n = \sum_{r\geq 0} \frac{q^r q^{r(r-1)/2} q^{r(r-1)/2}}{(q)_r} = \sum_{r\geq 0} \frac{q^{r^2}}{(q)_r}$$

$$\lim_{n\to\infty} B_n(q) = \lim_{n\to\infty} D_n - C_n = \sum_{r\geq 0} \frac{q^{2r} q^{r(r-1)/2} q^{r(r-1)/2}}{(q)_r} = \sum_{r\geq 0} \frac{q^{r^2+r}}{(q)_r}.$$

This proves the first equalities in each case of (3.10).

By the second equalities at (3.1) and (3.2),

$$\lim_{n\to\infty} A_n(q) = \lim_{n\to\infty} C_n$$

$$
\begin{aligned}
&= \frac{1}{(q)_\infty}\left(1+\sum_{r\geq 1}\frac{(1-q^{2r})(-1)^r(q)_{r-1}q^{5r^2/2-r/2}}{(q)_r}\right)\\
&= \frac{1}{(q)_\infty}\sum_{r=-\infty}^{\infty}(-q^{1/2})^r(q^{5/2})^{r^2}\\
&= \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q)_\infty}\\
&= \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} B_n(q) &= \lim_{n\to\infty} D_n - C_n\\
&= \frac{1}{(q^2)_\infty}\sum_{r\geq 0}\frac{(1-q^{2r+1})(-1)^r(q)_r q^{5r^2/2+3r/2}}{(1-q)(q)_r}\\
&= \frac{1}{(q^2)_\infty}\sum_{r=-\infty}^{\infty}(-q^{3/2})^r(q^{5/2})^{r^2}\\
&= \frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty}{(q)_\infty}\\
&= \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.
\end{aligned}
$$

This proves the second equalities in each case of (3.10). The penultimate equalities in each case above come from the Jacobi triple product identity. This is Hirschhorn's proof in [14].

However we desire to give an alternative proof for the infinite product identities so let $-a = b = q^{1/2}$, $c = 0$ and $d = 1$ in Theorem 3, so that the continued fraction in the theorem becomes $1/K(q^2)$. Thus, from the second equality in (3.1),

$$
\begin{aligned}
\lim_{n\to\infty} B_n(q^2) = \lim_{n\to\infty} C_n &= (-q^{3/2})_\infty(q^{3/2})_\infty\sum_{r\geq 0}\frac{q^{3r(r-1)/2+3r}}{(q^{3/2})_r(-q^{3/2})_r(q)_r}\\
&= (q;q^2)_\infty\sum_{r\geq 0}\frac{q^{3r(r+1)/2}}{(q;q^2)_{r+1}(q)_r}\\
&= (q;q^2)_\infty\frac{(q^8;q^{10})_\infty(q^2;q^{10})_\infty(q^{10};q^{10})_\infty}{(q;q)_\infty}\\
&= \frac{1}{(q^4;q^{10})_\infty(q^6;q^{10})_\infty}.
\end{aligned}
$$

For the third equality we have used (3.8). From the second equality in (3.1),

$$\begin{aligned}
\lim_{n\to\infty} D_n - C_n &= q^2(-q^{5/2})_\infty(q^{5/2})_\infty \sum_{r\geq 0} \frac{q^{3r(r-1)/2+5r}}{(q^{5/2})_r(-q^{5/2})_r(q)_r}\\
&= q^2(q;q^2)_\infty \sum_{r\geq 0} \frac{q^{3r(r-1)/2+5r}}{(q;q^2)_{r+2}(q)_r}\\
&= q^2(q;q^2)_\infty \sum_{r\geq 0} \frac{q^{3r(r-1)/2+5r}(1-q^{r+1})}{(q;q^2)_{r+2}(q)_{r+1}}\\
&= (q;q^2)_\infty \sum_{r\geq 1} \frac{q^{(3r^2+r)/2}(1-q^r)}{(q;q^2)_{r+1}(q)_r}\\
&= (q;q^2)_\infty \sum_{r\geq 0} \frac{q^{(3r^2+r)/2}(1-q^r)}{(q;q^2)_{r+1}(q)_r}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \lim_{n\to\infty} A_n(q^2) = \lim_{n\to\infty} D_n &= (q;q^2)_\infty \sum_{r\geq 0} \frac{q^{(3r^2+r)/2}}{(q;q^2)_{r+1}(q)_r}\\
&= (q;q^2)_\infty \sum_{r\geq 0} \frac{(-q;q)_r q^{(3r^2+r)/2}}{(q)_{2r+1}}\\
&= (q;q^2)_\infty \frac{(q^6;q^{10})_\infty(q^4;q^{10})_\infty(q^{10};q^{10})_\infty}{(q;q)_\infty}\\
&= \frac{1}{(q^2;q^{10})_\infty(q^8;q^{10})_\infty}.
\end{aligned}$$

The result now follows after replacing q^2 by q. □

4. A generalization of a continued fraction of Ramanujan

Before coming to our results in this section we need to state some notation and to recall some other necessary results.

We call $d_0 + K_{n=1}^{\infty} c_n/d_n$ a *canonical contraction* of $b_0 + K_{n=1}^{\infty} a_n/b_n$ if

$$C_k = A_{n_k}, \qquad D_k = B_{n_k} \qquad \text{for } k = 0,1,2,3,\dots,$$

where C_n, D_n, A_n and B_n are canonical numerators and denominators of $d_0 + K_{n=1}^{\infty} c_n/d_n$ and $b_0 + K_{n=1}^{\infty} a_n/b_n$ respectively. From [17] (page 85) we have:

Theorem 6. *The canonical contraction of* $b_0+K_{n=1}^{\infty} a_n/b_n$ *with* $C_0 = A_1/B_1$ *and*

$$C_k = A_{2k+1} \qquad D_k = B_{2k+1} \qquad \textit{for } k = 1,2,3,\dots,$$

exists if and only if $b_{2k+1} \neq 0 for K = 0,1,2,3,\dots$, *and in this case is given by*

$$\frac{b_0b_1+a_1}{b_1} - \frac{a_1a_2b_3/b_1}{b_1(a_3+b_2b_3)+a_2b_3} - \frac{a_3a_4b_5b_1/b_3}{a_5+b_4b_5+a_4b_5/b_3} - \frac{a_5a_6b_7/b_5}{a_7+b_6b_7+a_6b_7/b_5} - \frac{a_7a_8b_9/b_7}{a_9+b_8b_9+a_8b_9/b_7} + \cdots. \tag{4.1}$$

The continued fraction (4.1) is called the *odd* part of $b_0 + K_{n=1}^{\infty} a_n/b_n$.

We will also make use of the following theorem of Worpitzky (see [17], pp. 35–36).

Theorem 7. *(Worpitzky) Let the continued fraction $K_{n=1}^{\infty} a_n/1$ be such that $|a_n| \leq 1/4$ for $n \geq 1$. Then$K_{n=1}^{\infty} a_n/1$ converges. All approximants of the continued fraction lie in the disc $|w| < 1/2$ and the value of the continued fraction is in the disk $|w| \leq 1/2$.*

The following identity can be found in Ramanujan's notebooks [20] (a proof can be found in [10]).

Entry17 (p.374). Let a, b and q be complex numbers, with $|q| < 1$. Define

$$\phi(a) = \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2} a^n}{(q;q)_n(-bq;q)_n}.$$

Then

$$\frac{\phi(a)}{\phi(aq)} = 1 + \frac{aq}{1} + \frac{bq}{1} + \frac{aq^2}{1} + \frac{bq^2}{1} + \frac{aq^3}{1} + \frac{bq^3}{1} + \cdots. \tag{4.2}$$

The continued fraction in (4.2) is the special case $\lambda = 0$ of the following more general continued fraction ,which can be found in the lost notebook [19].

Set

$$F(a,b,\lambda,q) = 1 + \frac{aq+\lambda q}{1} + \frac{bq+\lambda q^2}{1} + \frac{aq^2+\lambda q^3}{1} + \frac{bq^2+\lambda q^4}{1} + \cdots$$

and

$$G(a,b,\lambda,q) = \sum_{n=0}^{\infty} \frac{(-\lambda/a)_n q^{(n^2+n)/2} a^n}{(q;q)_n(-bq;q)_n}.$$

Then

$$F(a,b,\lambda,q) = \frac{G(a,b,\lambda,q)}{G(aq,b,\lambda q,q)}. \tag{4.3}$$

A number of authors have given proofs of (4.3), including Hirschhorn [15].

We now evaluate two other continued fractions which also specialize to give the continued fraction in (4.2).

Corollary 5. *Let a, b, e and q be complex numbers with $|q| < 1$ and $|e| < 1/4$.*

(i) Set

$$\phi(x) = \sum_{j=0}^{\infty} \frac{\left(\frac{x}{e+1}\right)^j \left(\frac{eaq}{x(e+1)}\right)_j q^{j(j+1)/2}}{(q)_j \left(\frac{-bq}{e+1}\right)_j},$$

and define

$$H_2(a,b,e,q) := 1 + \frac{aq}{1} + \frac{bq+e}{1} + \frac{aq^2}{1} + \frac{bq^2+e}{1} + \frac{aq^3}{1} + \frac{bq^3+e}{1} + \cdots.$$

If $e \neq -bq^n$*, for* $n \geq 1$*, then*

$$H_2(a,b,e,q) = \frac{\phi(a)}{\phi(aq)}. \tag{4.4}$$

(ii) Set

$$\phi(x) = \sum_{j=0}^{\infty} \frac{\left(\frac{x}{e+1}\right)^j \left(\frac{eb}{a(e+1)}\right)_j q^{j(j+1)/2}}{(q)_j \left(\frac{-bq}{e+1}\right)_j},$$

and define

$$H_3(a,b,e,q) := 1 + \frac{aq+e}{1} + \frac{bq}{1} + \frac{aq^2+e}{1} + \frac{bq^2}{1} + \frac{aq^3+e}{1} + \frac{bq^3}{1} + \cdots.$$

If $e \neq -aq^n$*, for* $n \geq 1$*, then*

$$H_3(a,b,e,q) = (e+1)\frac{\phi(a)}{\phi(aq)}. \tag{4.5}$$

Remarks: (i) Note that the value $e = 0$ gives Ramanujan's identity (4.2) in each case.
(ii) The results may also hold for some e satisfying $|e| \geq 1/4$, but we do not explore that here.

Proof. (i) The condition on e ensures, by Worpitzky's Theorem, that the continued fraction above converges and hence equals its odd part, which is

$$\begin{aligned}
&1 + aq - \frac{aq(bq+e)}{aq^2+bq+e+1} - \frac{aq^2(bq^2+e)}{aq^3+bq^2+e+1} - \cdots \\
&\qquad - \frac{aq^{n+1}(bq^{n+1}+e)}{aq^{n+2}+bq^{n+1}+e+1} - \cdots \\
&= 1 + aq + \frac{(-aqb)q + (-aqe)}{(aq+b)q+e+1} + \frac{(-aqb)q^3 + (-aqe)q}{(aq+b)q^2+e+1} + \cdots \\
&\qquad + \frac{(-aqb)q^{2n+1} + (-aqe)q^n}{(aq+b)q^{n+1}+e+1} + \cdots.
\end{aligned}$$

Theorem 3 applied to the continued fraction $H_1(b, aq, -aqe, e+1, q)$ gives that

$$\frac{(-aqb)q+(-aqe)}{(aq+b)q+e+1}+\frac{(-aqb)q^3+(-aqe)q}{(aq+b)q^2+e+1}+\cdots$$
$$+\frac{(-aqb)q^{2n+1}+(-aqe)q^n}{(aq+b)q^{n+1}+e+1}+\cdots$$
$$=\frac{\dfrac{-aq(e+bq)}{e+1}\displaystyle\sum_{j=0}^{\infty}\frac{(a/(e+1))^j(e/(e+1))_j\,q^{j(j+5)/2}}{(q)_j(-bq/(e+1))_{j+1}}}{\displaystyle\sum_{j=0}^{\infty}\frac{(a/(e+1))^j(e/(e+1))_j\,q^{j(j+3)/2}}{(q)_j(-bq/(e+1))_j}}.$$

Thus

$$H_2(a,b,e,q)$$
$$=1+aq+\frac{\dfrac{-aq(e+bq)}{e+1}\displaystyle\sum_{j=0}^{\infty}\frac{(aq/(e+1))^j(e/(e+1))_j\,q^{j(j+3)/2}}{(q)_j(-bq/(e+1))_{j+1}}}{\displaystyle\sum_{j=0}^{\infty}\frac{(aq/(e+1))^j(e/(e+1))_j\,q^{j(j+1)/2}}{(q)_j(-bq/(e+1))_j}}.$$

Hence the result will be true if

$$(1+aq)\sum_{j=0}^{\infty}\frac{(aq/(e+1))^j(e/(e+1))_j\,q^{j(j+1)/2}}{(q)_j(-bq/(e+1))_j}$$
$$-\frac{aq(e+bq)}{e+1}\sum_{j=0}^{\infty}\frac{(aq/(e+1))^j(e/(e+1))_j\,q^{j(j+3)/2}}{(q)_j(-bq/(e+1))_{j+1}}$$
$$=\sum_{j=0}^{\infty}\frac{(a/(e+1))^j\,(eq/(e+1))_j\,q^{j(j+1)/2}}{(q)_j\,(-bq/(e+1))_j}.$$

This follows easily by considering both sides as power series in a and comparing coefficients. The coefficient of a^0 is clearly seen to be 1 on each side. For $j\geq 1$, the coefficient of a^j on the left side is

$$\frac{(q/(e+1))^j(e/(e+1))_j\,q^{j(j+1)/2}}{(q)_j(-bq/(e+1))_j}$$
$$+q\frac{(q/(e+1))^{j-1}(e/(e+1))_{j-1}\,q^{j(j-1)/2}}{(q)_{j-1}(-bq/(e+1))_{j-1}}$$
$$-\frac{q(e+bq)}{e+1}\frac{(q/(e+1))^{j-1}(e/(e+1))_{j-1}\,q^{(j-1)(j+2)/2}}{(q)_{j-1}(-bq/(e+1))_j}$$
$$=\frac{(1/(e+1))^j(e/(e+1))_{j-1}\,q^{j(j+1)/2}}{(q)_j(-bq/(e+1))_j}\times\left(\left(1-\frac{eq^{j-1}}{e+1}\right)q^j\right.$$
$$\left.+(e+1)\left(1-q^j\right)\left(1+\frac{bq^j}{e+1}\right)-(e+bq)\left(1-q^j\right)q^{j-1}\right)$$

$$= \frac{(1/(e+1))^j (eq/(e+1))_j \, q^{j(j+1)/2}}{(q)_j(-bq/(e+1))_j}.$$

(ii) The proof in this case is initially similar to that of (i). We first show, using similar reasoning to that in part (i), that

$$H_3(a,b,e,q) = \frac{(e+1)\displaystyle\sum_{j=0}^{\infty} \frac{(b/(e+1))^j (e/(e+1))_j \, q^{j(j+1)/2}}{(q)_j(-aq/(e+1))_j}}{\dfrac{1}{1+aq/(e+1)} \displaystyle\sum_{j=0}^{\infty} \frac{(b/(e+1))^j (e/(e+1))_j \, q^{j(j+1)/2}}{(q)_j(-aq^2/(e+1))_j}}.$$

The identity at (3.4) is now applied to the numerator and denominator in this last expression and (4.5) follows. □

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