Some Remarks on the Coefficients of Hecke Eigenforms and Chebyshev Polynomials of the Second Kind

West Coast Number Theory, 2022

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James Mc Laughlin
(Joint work with Tim Huber, Larry Rolen and Dongxi Ye

West Chester University, PA

http://www.wcupa.edu/sciences-mathematics/mathematics/jmclaughlin/

jmclaughlin2@wcupa.edu

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Overview

Background and Notation

q-products

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For $|q| < 1$, $$(q;q)_\infty := (1-q)(1-q^2)(1-q^3)\cdots$$

$$f_1 := (q;q)_\infty \qquad f_j := (q^j;q^j)_\infty$$

The series $\sum_{n=0}^\infty c(n)q^n$ is *lacunary* if

$$\lim_{x\to\infty} \frac{|\{0 \le n \le x \mid c(n) = 0\}|}{x} = 1.$$

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Serre: for even positive integers s, f_1^s is lacunary if and only if

$$s \in \{2, 4, 6, 8, 10, 14, 26\}.$$

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An *eta quotient* is a finite product of the form $\prod_j f_j^{n_j}$, for some integers $j \in \mathbb{N}$ and $n_j \in \mathbb{Z}$.

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Theorem

(Han and Ono, 2011) Assuming the notation above, we have that

$$a(n) = 0 \iff b(n) = 0. \tag{2}$$

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Moreover, we have that $a(n) = b(n) = 0$ *precisely for those non-negative* n *for which* $\mathrm{ord}_p(3n+1)$ *is odd for some prime* $p \equiv 2 \pmod 3$.

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What was discovered as a result of these computer algebra experiments is summarized as follows.

Other eta quotients with identically vanishing coefficients I

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Let $(A(q), B(q))$ be any of the pairs

$$\left\{ \left(f_1^4, \frac{f_1^8}{f_2^2}\right), \left(f_1^4, \frac{f_1^{10}}{f_3^2}\right), \left(f_1^6, \frac{f_2^4}{f_1^2}\right), \left(f_1^6, \frac{f_1^{14}}{f_2^4}\right), \left(f_1^{10}, \frac{f_2^6}{f_1^2}\right), \left(f_1^{14}, \frac{f_3^5}{f_1}\right), \left(f_1^{14}, \frac{f_2^8}{f_1^2}\right) \right\}. \quad (3)$$

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For any such pair $(A(q), B(q))$, define the sequences $\{a(n)\}$ and $\{b(n)\}$ by

$$A(q) =: \sum_{n=0}^{\infty} a(n) q^n, \qquad B(q) =: \sum_{n=0}^{\infty} b(n) q^n. \quad (4)$$

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Then, for each pair, $a(n) = 0 \iff b(n) = 0$, with criteria for when exactly this happens.

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Aside: The results above on identically vanishing coefficients appear to be just “the tip of the iceberg”.

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- Use the multiplicativity of the coefficients in the CM forms, and the recursive formula for prime powers (more on these later) to determine information about a general coefficient b_n (and in particular, when $b_n = 0$).

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While trying to prove the (possibly false) reverse direction, the speaker was led to the result described in the next few slides.

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$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x). \tag{7}$$

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Let $p \nmid N$ be a prime, so that the following recurrence formula holds

$$a_{p^{n+1}} = a_{p^n} a_p - \chi(p) p^{k-1} a_{p^{n-1}}. \tag{8}$$

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Then, after fixing a value for $\sqrt{\chi(p)}$*,*

$$a_{p^n} = \left(-p^{(k-1)/2}\sqrt{\chi(p)}\right)^n U_n\left(\frac{-a_p}{2p^{(k-1)/2}\sqrt{\chi(p)}}\right). \tag{9}$$

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so that (10) becomes $u_{n+1} = 2x\, u_n - u_{n-1}$,

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$x = -a_p/(2p^{(k-1)/2}\sqrt{\chi(p)})$, so that $u_1 = 2x$ and $u_0 = a_1 = 1$. □

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(2) Known results about Chebyshev polynomials of the second kind can now be used to derive various identities for terms in the sequence $\{a_{p^n}\}$, where p is a prime.

Properties of Chebyshev polynomials of the second kind

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Applications to the Fourier Coefficients of Hecke Eigenforms

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Application of identities for Chebyshev polynomials of the second kind I

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Let $f(q) = q + \sum_{n=2}^{\infty} a_n q^n$ be a normalized Hecke eigenform of weight k, level N, and Nebentypus χ.

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Application of identities for Chebyshev polynomials of the second kind I

Let $f(q) = q + \sum_{n=2}^{\infty} a_n q^n$ be a normalized Hecke eigenform of weight k, level N, and Nebentypus χ. Let $p \nmid N$ be a prime. .
The identities in the previous section are used in conjunction with the identity

$$a_{p^n} = \left(-p^{(k-1)/2}\sqrt{\chi(p)}\right)^n U_n\left(\frac{-a_p}{2p^{(k-1)/2}\sqrt{\chi(p)}}\right), \tag{26}$$

to derive identities for the members of the sequence $\{a_{p^n}\}$.

Application of identities for Chebyshev polynomials of the second kind II

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These general identities mentioned on the previous slide are illustrated using the Ramanujan τ function, defined by

$$q\prod_{m=1}^{\infty}(1-q^m)^{24} =: \sum_{n=1}^{\infty}\tau(n)q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + 84480q^8 - 113643q^9 - 115920q^{10} + 534612q^{11} - 370944q^{12} - 577738q^{13} + 401856q^{14} + 1217160q^{15} + 987136q^{16} - \dots$$

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Recall: (1) $\tau(m)\tau(n) = \tau(mn)$ if $\gcd(m,n) = 1$.

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Recall: (1) $\tau(m)\tau(n) = \tau(mn)$ if $\gcd(m,n) = 1$.
For example, $\tau(3)\tau(5) = 252 \times 4830 = 1217160 = \tau(15)$.

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$$L(f,s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \sum_{n=0}^{\infty} \frac{a_{p^n}}{p^{sn}} = \prod_p \frac{1}{1-a_p p^{-s} + \chi(p) p^{-2s} p^{k-1}}. \tag{29}$$

An L-function for the sequence $\{a_n^2\}$

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$$\sum_{n=0}^{\infty} U_n^2(x) t^n = \frac{(t+1)}{(1-t)\left((t+1)^2 - 4tx^2\right)} \tag{30}$$

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For convergence we may take $Re(s) > k$.

Ramanujan τ-function, Example I

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Example

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$$\sum_{n=0}^{\infty} \frac{a_{p^n} t^n}{n!} = \exp\left(\frac{a_p t}{2}\right) \left(\cos\left(\frac{1}{2} t \sqrt{4p^{k-1}\chi(p) - a_p^2}\right) \right.$$
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$$\sum_{n=0}^{\infty} \frac{\tau\left(p^n\right) t^{n+1}}{(n+1)!} = \frac{2e^{\frac{t\tau(p)}{2}} \sin\left(\frac{1}{2} t \sqrt{4p^{11} - \tau(p)^2}\right)}{\sqrt{4p^{11} - \tau(p)^2}}.$$

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$$F_{\pm} = a_{p_1} a_{p_2} \pm \sqrt{4p_1^{k-1}\chi(p_1) - a_{p_1}^2}\sqrt{4p_2^{k-1}\chi(p_2) - a_{p_2}^2},$$

$$\Phi_{\pm} = a_{p_1}\sqrt{4p_2^{k-1}\chi(p_2) - a_{p_2}^2} \pm a_{p_2}\sqrt{4p_1^{k-1}\chi(p_1) - a_{p_1}^2}.$$

Then for any $t \in \mathbb{C}$,

Identities from the Bivariate Generating Functions I

From the bivariate generating functions at (24) and (25):

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Then for any $t \in \mathbb{C}$,

$$\sum_{n=0}^{\infty} a_{p_1^n} a_{p_2^n} \frac{t^{n+1}}{(n+1)!} = 2\frac{e^{t/4F_+}\cos(t/4\Phi_-) - e^{t/4F_-}\cos(t/4\Phi_+)}{\sqrt{4p_1^{k-1}\chi(p_1) - a_{p_1}^2}\sqrt{4p_2^{k-1}\chi(p_2) - a_{p_2}^2}}. \quad (34)$$

Identities from the Bivariate Generating Functions II

Identities from the Bivariate Generating Functions II

Theorem (continued)

For any $t \in \mathbb{C}$ *satisfying* $|t| < (p_1 p_2)^{-k/2}$,

Identities from the Bivariate Generating Functions II

Theorem (continued)

For any $t \in \mathbb{C}$ *satisfying* $|t| < (p_1 p_2)^{-k/2}$,

$$\sum_{n=0}^{\infty} a_{p_1^n} a_{p_2^n} t^n$$
$$= \frac{1 - t^2 p_1^{k-1} p_2^{k-1} \chi(p_1)\chi(p_2)}{\left(1 - t^2 p_1^{k-1} p_2^{k-1} \chi(p_1)\chi(p_2)\right)^2 - t\left(a_{p_1} - t a_{p_2} p_1^{k-1}\chi(p_1)\right)\left(a_{p_2} - t a_{p_1} p_2^{k-1}\chi(p_2)\right)}. \tag{35}$$

Ramanujan τ-function, Example III

Ramanujan τ-function, Example III

Example

Let p_1 and p_2 be primes (distinct or otherwise) and define

$$F_\pm = \tau(p_1)\tau(p_2) \pm \sqrt{4p_1^{11} - \tau^2(p_1)}\sqrt{4p_2^{11} - \tau^2(p_2)},$$
$$\Phi_\pm = \tau(p_1)\sqrt{4p_2^{11} - \tau^2(p_2)} \pm \tau(p_2)\sqrt{4p_1^{11} - \tau^2(p_1)}.$$

Ramanujan τ-function, Example III

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Then for any $t \in \mathbb{C}$,

$$\sum_{n=0}^{\infty} \tau(p_1^n)\tau(p_2^n)\frac{t^{n+1}}{(n+1)!} = 2\frac{e^{t/4F_+}\cos(t/4\Phi_-) - e^{t/4F_-}\cos(t/4\Phi_+)}{\sqrt{4p_1^{11} - \tau^2(p_1)}\sqrt{4p_2^{11} - \tau^2(p_2)}}. \quad (36)$$

Ramanujan τ-function, Example III Continued

Ramanujan τ-function, Example III Continued

Example (continued)

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For any $t \in \mathbb{C}$ satisfying $|t| < (p_1 p_2)^{-6}$,

Ramanujan τ-function, Example III Continued

Example (continued)

For any $t \in \mathbb{C}$ satisfying $|t| < (p_1 p_2)^{-6}$,

$$\sum_{n=0}^{\infty} \tau(p_1^n)\tau(p_2^n)t^n$$

$$= \frac{1 - p_1^{11} p_2^{11} t^2}{\left(1 - p_1^{11} p_2^{11} t^2\right)^2 - t\left(\tau\left(p_1\right) - p_1^{11}\tau\left(p_2\right)t\right)\left(\tau\left(p_2\right) - p_2^{11}\tau\left(p_1\right)t\right)}.$$

An Identity Implying a Divisibility Property of the Sequence a_{p^n}

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Theorem

Let the sequence a_{p^n} be as defined in Proposition 2.1.

An Identity Implying a Divisibility Property of the Sequence a_{p^n}

Theorem

Let the sequence a_{p^n} be as defined in Proposition 2.1. If $m \geq 1$ and $n \geq 2$ are integers, then

$$a_{p^{mn-1}} = a_{p^{n-1}} \times \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} (-1)^j \binom{m-1-j}{j} \left(a_{p^n} - p^{k-1}\chi(p) a_{p^{n-2}} \right)^{m-1-2j} p^{(k-1)nj} \chi^j(p). \tag{37}$$

An Identity Implying a Divisibility Property of the Sequence a_{p^n}

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Remark: Note that if the numbers a_{p^n} are integers, then (37) implies that if $n+1 | m+1$, then $a_{p^n} | a_{p^m}$.

Ramanujan τ-function, Example IV

Ramanujan τ-function, Example IV

Example

If $m \geq 1$ and $n \geq 2$ are integers, then

$$\tau(p^{mn-1}) = \tau(p^{n-1}) \times \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} (-1)^j \binom{m-1-j}{j} \left(\tau(p^n) - p^{11}\tau(p^{n-2})\right)^{m-1-2j} p^{11nj}.$$

Ramanujan τ-function, Example IV

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If m and n are positive integers such that $n+1|m+1$,

Ramanujan τ-function, Example IV

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If m and n are positive integers such that $n+1 | m+1$, then

$$\tau(p^n) | \tau(p^m).$$

Ramanujan τ-function, Example IV

Example

If $m \geq 1$ and $n \geq 2$ are integers, then

$$\tau(p^{mn-1}) = \tau(p^{n-1}) \times \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} (-1)^j \binom{m-1-j}{j} \left(\tau(p^n) - p^{11}\tau(p^{n-2})\right)^{m-1-2j} p^{11nj}.$$

If m and n are positive integers such that $n+1 | m+1$, then

$$\tau(p^n) | \tau(p^m).$$

For example, taking $m = 119$ and considering the divisors of 120, then for any prime p,

$$\tau(p^n) | \tau(p^{119}) \text{ for any } n \in \{1, 2, 3, 4, 5, 7, 9, 11, 14, 19, 23, 29, 39, 59\}.$$

Thanks

Thank you for listening/watching.