# THE TOPOLOGY OF SHRINKING WEDGES AND RELATED CONSTRUCTIONS

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ABSTRACT. This note is dedicated to identifying basic topological equivalences related to combinations of shrinking wedges (such as the $n$-dimensional earring), smash products, and infinite direct products.

## 1. INTRODUCTION AND REVIEW OF STANDARD CONSTRUCTIONS

This note contains explicit proofs of basic equivalences and facts involving operations on spaces that are important within wild algebraic topology. Certainly, no originality is claimed as most results are fairly straightforward to prove. It is possible that much of this is "folklore." However, this note was a consequence of taking the time to check that these results are true.

Shrinking wedges of spaces, like the earring space, play a prominent role in the homotopy theory of Peano continua and other "wild" spaces that may not have the homotopy type of a CW-complex. Standard constructions on spaces such a products, wedges (one-point unions), suspensions, cones, smash products, and various combinations of these spaces play an important role in ordinary algebraic topology [3]. This note is dedicated to identifying some properties of their infinitary analogues that are relevant to the progressive theory of "wild algebraic topology." Here the word "infinitary" refers to an operation with infinitely many inputs. Such constructions include infinite direct products, shrinking wedges, and an infinite analogue of the smash product.

**Definition 1.1** (One-point Unions)**.** The *wedge* of a family $\{(X_i, x_i) \mid i \in I\}$ of based spaces is the space $\bigvee_{i\in I} X_i = \coprod_{i\in I} X_i/\sim$ where the set $\{x_i \mid i \in I\}$ is identified to a single point $b_0$. The point $b_0$ is the basepoint; we sometimes refer to it as the *wedge-point.* A set $U \subseteq \bigvee_{i\in I} X_i$ is open if and only if $U \cap X_i$ is open in $X_i$ for all $i \in I$. In other words, $\bigvee_{i\in I} X_i$ has the weak topology with respect to the family of subspaces $\{X_i \mid i \in I\}$. This operation serves as the coproduct in the category of based spaces. In particular, there are canonical embeddings $X_i \to \bigvee_{i\in I} X_i$ for each $i$, which map $X_i$ onto the $i$-th "summand."

**Definition 1.2** (Direct Products)**.** The *direct product* of a family $\{X_i \mid i \in I\}$ of based spaces is the space $\prod_{i\in I} X_i$ consisting of $I$-tuples $(x_i)_{i\in I}$ with $x_i \in X_i$, which formally are choice functions $\mathbf{x} : I \to \coprod_{i\in I} X_i$ such that $\mathbf{x}(i) \in X_i$ for all $i \in I$. When $a_i \in X_i$ is a given basepoint, we take $(a_i)_{i\in I}$ to be the basepoint of the product. We will give this space the product topology unless stated otherwise. If $I = \{1, 2, \ldots, n\}$, we may write this product as $\prod_{i=1}^{n} X_i$ or $X_1 \times X_2 \times \cdots \times X_n$.

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**Remark 1.3.** For a finite collection $\{(X_i, x_i) \mid 1 \leqslant i \leqslant n\}$ of based spaces, we identify the wedge $\bigvee_{i=1}^n X_i$ canonically as a subspace of $\prod_{i=1}^n X_i$ where the $i$-th summand of the wedge is identified with the subspace $\{x_1\} \times \{x_2\} \times \cdots \times \{x_{i-1}\} \times X_i \times \{x_{i+1}\} \times \cdots \times \{x_n\}$ of the product.

**Definition 1.4** (Smash Products)**.** The *smash product* of two based spaces $(X, x_0)$ and $(Y, y_0)$ is the quotient space $X \wedge Y = X \times Y / X \vee Y$, where we identify the wedge $X \vee Y$ is identified with $X \times \{y_0\} \cup \{x_0\} \times Y$ as mentioned above. The image of $X \vee Y$ in the quotient is the basepoint of $X \wedge Y$. As special cases:

(1) if $(S^n, e_n)$ is the based $n$-sphere, then $\Sigma^n X = X \wedge S^n$ is the *$n$-th reduced suspension* of $(X, x_0)$.
(2) if $[0, 1]$ is the closed unit interval with basepoint 0, then $C_* X = X \wedge [0, 1]$ is the *reduced cone* of $(X, x_0)$.

## 2. Shrinking wedges

A shrinking wedge of a countably infinite family of based spaces is exactly what it sounds like. The underlying set is the same as the usual one-point union but we give it a coarser topology so that the summands "shrink" toward the wedge-point.

**Definition 2.1** (Shrinking Wedge)**.** The *shrinking wedge* of countable set $\{(X_j, x_j)\}_{j\in J}$ of based spaces is space $\widetilde{\bigvee}_{j\in J}(X_j, x_j)$ whose underlying set is the usual one-point union $\widetilde{\bigvee}_{j\in J}(X_j, x_j)$ with canonical basepoint $b_0$. A set $U$ is open in $\widetilde{\bigvee}_{j\in J} X_j$ if $U \cap X_j$ is open in $X_j$ for all $j \in J$ and if $X_j \subseteq U$ for all but finitely many $j \in J$ whenever $b_0 \in U$. When the basepoints are clear from context, we may write the shrinking wedge as $\widetilde{\bigvee}_{j\in J} X_j$ and if $X_j = Z$ for all $j \in J$, we may denote it as $\widetilde{\bigvee}_J Z$.

**Example 2.2.** The special case $\mathbb{E}_n = \widetilde{\bigvee}_{\mathbb{N}} S^n$ is called the *$n$-dimensional earring.* It is known that $\mathbb{E}_n$ is $(n-1)$-connected and $\pi_n(\mathbb{E}_n) \cong \mathbb{Z}^{\mathbb{N}}$; see [1].

**Remark 2.3.** Let $\{(X_i, x_i)\}_{i\in\mathbb{N}}$ be a sequence of based spaces. We may consider the shrinking wedge $\widetilde{\bigvee}_{i\in\mathbb{N}} X_i$ naturally as a subspace of the infinite direct product $\prod_{i\in\mathbb{N}} X_i$ where $X_i$ is identified with

$$\{x_1\} \times \{x_2\} \times \cdots \times X_{i-1} \times X_i \times \{x_{i+1}\} \times \cdots$$

and the basepoint is $b_0 = (x_1, x_2, x_3, \dots)$.

**Lemma 2.4.** *If $\{X_i \mid i \in I\}$ is a countably infinite collection of based spaces and $F \subseteq I$ is a finite subset, then*

$$\widetilde{\bigvee}_{i\in I} X_i \cong \left(\bigvee_{i\in F} X_i\right) \vee \left(\widetilde{\bigvee}_{i\in I\setminus F} X_i\right).$$

**Lemma 2.5.** *If $\{X_{i,j} \mid (i,j) \in I \times J\}$ is a countable collection of based spaces, then*

$$\widetilde{\bigvee}_{(i,j)\in I\times J} X_{i,j} \cong \widetilde{\bigvee}_{i\in I}\left(\widetilde{\bigvee}_{j\in J} X_{i,j}\right).$$

**Remark 2.6.** Many topological properties pass from the spaces $X_i$ to the shrinking wedge $\widetilde{\bigvee}_{i\in I} X_i$. For instance, if each $X_i$ has one of the following properties, then so does $\widetilde{\bigvee}_{i\in I} X_i$: Hausdorff, regular, completely regular compact, metrizable, first countable, second countable, path-connected, path-connected and locally path-connected (as a combined property), among others. With some exceptions, verifying these properties often amounts to observing that the property in question is closed under forming countable direct products and closed subspaces (since $\widetilde{\bigvee}_{i\in I}X_i \subseteq \prod_{i\in I} X_i$).

To prove general facts about shrinking wedges, it is convenient to consider the following unbased analogue.

**Definition 2.7.** Given a countable (not necessarily ordered) collection of spaces $\{X_i \mid i \in I\}$, the *shrinking disjoint union* of this collection is the space $\widetilde{\coprod}_{i\in I}X_i$ whose underlying set is $\{x_0\} \cup \coprod_{i\in I} X_i$ where $x_0$ is an added point. A set $U$ is open in $\widetilde{\coprod}_{i\in I}X_i$ if and only if $U \cap X_i$ is open in $X_i$ for all $i \in I$ and $X_i \subseteq U$ for all but finitely many $i$ whenever $x_0 \in U$. We refer to $x_0$ as the *primary limit point* of $\widetilde{\coprod}_{i\in I}X_i$.

Note that the subspace $\coprod_{i\in I} X_i$ of $\widetilde{\coprod}_{i\in I}X_i$ has the usual disjoint union topology. We usually will consider $\widetilde{\coprod}_{i\in I}X_i$ as a based space with basepoint $x_0$. By enumerating $I$, we may think of $\widetilde{\coprod}_{i\in I}X_i$ as a sequence of disjoint spaces, "converging to" $x_0$.

**Proposition 2.8.** *If $\{(X_i, x_i) \mid i \in I\}$ is a countable collection of based spaces, then the canonical map $\widetilde{\coprod}_{i\in I}X_i \to \widetilde{\bigvee}_{i\in I}X_i$ identifying $\{x_0\} \cup \{x_i \mid i \in I\}$ to the wedge point $b_0$ is a quotient map.*

*Proof.* Let $f$ denote the map $\widetilde{\coprod}_{i\in I}X_i \to \widetilde{\bigvee}_{i\in I}X_i$ in question. Consider $U \subseteq \widetilde{\bigvee}_{i\in I}X_i$ such that $f^{-1}(U)$ is open. If $b_0 \notin U$, then $f$ maps $f^{-1}(U)$ bijectively onto $U$ and $f^{-1}(U) = \coprod_{i\in I} X_i \cap U$ where $X_i \cap U$ is open in $X_i$ for each $i$. Thus $U$ is open in $\widetilde{\bigvee}_{i\in I}X_i$. If $b_0 \in U$, then we have $f^{-1}(b_0) = \{x_0\} \cup \{x_i \mid i \in I\} \subseteq f^{-1}(U)$. In particular, $U \cap X_i$ is an open neighborhood of $x_i$ in $X_i$. Moreover, since $f^{-1}(U)$ is open, we have $X_i \subseteq f^{-1}(U)$ for all but finitely many $i \in I$. We conclude that $U \cap X_i$ is open in $X_i$ for all $i$ and that $X_i \subseteq U$ for all but finitely many $i \in I$. $\square$

**Remark 2.9.** If $X_i^+ = X_i \cup \{x_i\}$ consists of the disjoint union of a space $X_i$ and an isolated basepoint $x_i$, then $\widetilde{\bigvee}_{i\in I}X_i^+ \cong \widetilde{\coprod}_{i\in I}X_i$.

The following lemma is straightforward to prove.

**Lemma 2.10.** *If $q_i : X_i \to Y_i$, $i \in I$ is a countable collection of quotient maps, then the induced based map $\widetilde{\coprod}_i q_i : \widetilde{\coprod}_{i\in I}X_i \to \widetilde{\coprod}_{i\in I}Y_i$ is a quotient map.*

*Proof.* Let $q = \widetilde{\coprod}_i q_i$ and suppose $U \subseteq \widetilde{\coprod}_{i\in I}Y_i$ such that $q^{-1}(U)$ is open. Since $q^{-1}(U) \cap \coprod_{i\in I} X_i = \coprod_{i\in I} q_i^{-1}(U \cap Y_i)$ is open, $q_i^{-1}(U \cap Y_i)$ is open in $X_i$ for all $i$. Since $q_i$ is quotient, $U \cap Y_i$ is open in $Y_i$ for all $i$. Moreover, if $y_0 \in U$, then $x_0 \in q^{-1}(U)$. In this case, we have $X_i \subseteq q^{-1}(U)$ for all but finitely many $i$ and thus $Y_i \subseteq U$ for all but finitely many $i$. We conclude that $U$ is open in $\widetilde{\coprod}_{i\in I}Y_i$. $\square$

**Corollary 2.11.** *If $q_i : (X_i, x_i) \to (Y_i, y_i)$, $i \in I$ is a countable collection of based quotient maps, then the induced map $\widetilde{\bigvee}_{i\in I}q_i : \widetilde{\bigvee}_{i\in I}X_i \to \widetilde{\bigvee}_{i\in I}Y_i$ is a quotient map.*

*Proof.* Consider the following commutative diagram. The vertical maps are quotient by Proposition 2.8 and the top map is quotient by Lemma 2.10.

$$\begin{tikzcd} \widetilde{\coprod}_{i\in I} X_i \arrow[r, "\widetilde{\coprod}_{i\in I} q_i"] \arrow[d] & \widetilde{\coprod}_{i\in I} X_i \arrow[d] \\ \widetilde{\bigvee}_{i\in I} X_i \arrow[r, "\widetilde{\bigvee}_{i\in I} q_i"'] & \widetilde{\bigvee}_{i\in I} Y_i \end{tikzcd}$$

It follows from the universal property of quotient maps that the bottom map must be quotient. $\square$

**Lemma 2.12.** *If $x_0 \in A \subseteq \widetilde{\bigvee}_{i\in I} X_i$, and $A_i = A \cap X_i$ for each $i \in I$, then $A \cong \widetilde{\bigvee}_{i\in I} A_i$ and*

$$\widetilde{\bigvee}_{i\in I} X_i \Big/ A \cong \widetilde{\bigvee}_{i\in I} (X_i/A_i).$$

*Proof.* Since $A$ has the subspace topology inherited from $\widetilde{\bigvee}_{i\in I} A_i$, every neighborhood of the wedgepoint $b_0$ in $\widetilde{\bigvee}_{i\in I} A_i$ contains $A_i$ for all but finitely many $i$. Hence $A \cong \widetilde{\bigvee}_{i\in I} A_i$. Let $q_i : X_i \to X_i/A_i$ be the quotient map collapsing $A_i$ to a point. We have the following commutative diagram where the vertical map $f$ is the quotient map collapsing $A$ to a point and the diagonal map $\widetilde{\bigvee}_{i\in I} q_i$ is quotient by Corollary 2.11.

$$\begin{tikzcd} \widetilde{\bigvee}_{i\in I} X_i \arrow[d, "f"'] \arrow[dr, "\widetilde{\bigvee}_{i\in I} q_i"] & \\ \widetilde{\bigvee}_{i\in I} X_i \Big/ A \arrow[r, "h"'] & \widetilde{\bigvee}_{i\in I} (X_i/A_i) \end{tikzcd}$$

The bottom map $h$ is an induced bijection since $f$ and $\widetilde{\bigvee}_{i\in I} q_i$ have identical fibers; most importantly, the fiber of the respective wedgepoints under these maps is $\widetilde{\bigvee}_{i\in I} A_i$. Since $f$ and $\widetilde{\bigvee}_{i\in I} q_i$ are quotient maps, $h$ is a homeomorphism. $\square$

A continuous surjection $f : A \to B$ is a *biquotient map* if for every open cover $\mathscr{U}$ of a fiber $f^{-1}(b)$ then there exists finitely many sets $U_1, U_2, \ldots, U_r \in \mathscr{U}$ such that $f(U_1 \cup U_2 \cup \cdots \cup U_n)$ contains a neighborhood of $b$. The notion of biquotient map is due to E. Michael [2] and is well-known in general topology. In particular, Michael showed that every biquotient map is quotient and arbitrary products of biquotient maps are biquotient. It is easy to see that the natural quotient map $\widetilde{\coprod}_{i\in I} X_i \to \widetilde{\bigvee}_{i\in I} X_i$ is, in fact, biquotient since the fiber of the wedge point is the compact set $\{x_0\} \cup \{x_i \mid i \in I\}$, which is homeomorphic to $\omega + 1$. We have the following consequence.

**Lemma 2.13.** *Given two countable collections $\{X_i \mid i \in I\}$ and $\{Y_j \mid j \in J\}$, the natural product map $\widetilde{\coprod}_{i\in I} X_i \times \widetilde{\coprod}_{j\in J} Y_j \to \widetilde{\bigvee}_{i\in I} X_i \times \widetilde{\bigvee}_{j\in J} Y_j$ is a quotient map.*

**Theorem 2.14.** *Given any two countable collections of based spaces $\{X_i \mid i \in I\}$ and $\{Y_j \mid j \in J\}$, there is a canonical homeomorphism $\widetilde{\bigvee}_{i\in I} X_i \wedge \widetilde{\bigvee}_{j\in J} Y_j \cong \widetilde{\bigvee}_{(i,j)\in I\times J} X_i \wedge Y_j$.*

*Proof.* Let $x_0, y_0, (x_0, y_0)$ denote the respective primarily limit points of the shrinking disjoint unions $\widetilde{\coprod}_{i\in I} X_i$, $\widetilde{\coprod}_{j\in J} Y_j$, and $\widetilde{\coprod}_{(i,j)\in I\times J} X_i \times Y_j$. Consider the following commutative diagram

$$\begin{tikzcd}
\widetilde{\coprod}_{i\in I} X_i \times \widetilde{\coprod}_{j\in J} Y_j \arrow[r, "q"] \arrow[d, "(1)"'] & \widetilde{\coprod}_{(i,j)\in I\times J} X_i \times Y_j \arrow[d, "(3)"] \\
\widetilde{\bigvee}_{i\in I} X_i \times \widetilde{\bigvee}_{j\in J} Y_j \arrow[d, "(2)"'] & \widetilde{\coprod}_{(i,j)\in I\times J} X_i \wedge Y_j \arrow[d, "(4)"] \\
\widetilde{\bigvee}_{i\in I} X_i \wedge \widetilde{\bigvee}_{j\in J} Y_j \arrow[r, dashed] & \widetilde{\bigvee}_{(i,j)\in I\times J} X_i \wedge Y_j
\end{tikzcd}$$

The map (1) is quotient by Lemma 2.13 and map (2) is quotient by the construction of the smash product. Map map (3) is quotient by Lemma 2.10 and map (4) is the quotient map from Proposition 2.8. Finally the horizontal map $q$ is the map that sends $A = \left(\{x_0\} \times \widetilde{\coprod}_j Y_j\right) \cup \left(\widetilde{\coprod}_i X_i \times \{y_0\}\right)$ to the limit point $\{(x_0, y_0)\}$ of $\widetilde{\coprod}_{(i,j)} X_i \times Y_j$ (notice that this is homeomorphic to $\widetilde{\coprod}_{i\in I} X_i \wedge \widetilde{\coprod}_{j\in J} Y_j$) and is the identity elsewhere. Since every neighborhood of $A$ in

$$\widetilde{\coprod}_{i\in I} X_i \times \widetilde{\coprod}_{j\in J} Y_j$$

contains $X_i \times Y_j$ for all but finitely many $(i,j) \in I \times J$, it follows that $q$ is a quotient map. By carefully considering the fibers of the map (1), one can check the the fibers of both compositions include the set $B = A \cup \coprod_{(i,j)\in I\times J} X_i \vee Y_j$ and singletons $\{(x,y)\}$ for points $(x,y) \notin B$. We conclude that there is an induced homeomorphism between the quotient spaces. □

**Example 2.15** (Suspensions of shrinking wedges)**.** For $n \in \mathbb{N}$, the $n$-th reduced suspension of a based space may be defined as $\Sigma^n X \cong X \wedge S^n$. If we apply Theorem 2.14 in the case $J = \mathbb{N}$, $Y_1 = S^n$, and $Y_j$ is a single-point space for $j \geqslant 2$, then we have canonical homeomorphisms

$$\Sigma^n \left(\widetilde{\bigvee}_{j\in\mathbb{N}} X_j\right) = \left(\widetilde{\bigvee}_{j\in\mathbb{N}} X_j\right) \wedge S^n \cong \widetilde{\bigvee}_{j\in\mathbb{N}} (X_j \wedge S^n) \cong \widetilde{\bigvee}_{j\in\mathbb{N}} \Sigma^n X_j.$$

In particular, $\Sigma^n \mathbb{E}_m \cong \mathbb{E}_{m+n}$ for all $m, n \geqslant 0$.

**Example 2.16** (Smash products of higher earrings)**.** It is well-known that $S^m \wedge S^n \cong S^{m+n}$. Consequently if $\mathbb{E}_m$ and $\mathbb{E}_n$ are the $m$-dimensional and $n$-dimensional earrings, then

$$\begin{aligned}
\mathbb{E}_m \wedge \mathbb{E}_n &= \left(\widetilde{\bigvee}_{i\in\mathbb{N}} S^m\right) \wedge \left(\widetilde{\bigvee}_{j\in\mathbb{N}} S^n\right) \\
&\cong \widetilde{\bigvee}_{(i,j)\in\mathbb{N}^2} (S^m \wedge S^n) \\
&\cong \widetilde{\bigvee}_{(i,j)\in\mathbb{N}^2} S^{m+n} \\
&\cong \mathbb{E}_{m+n}
\end{aligned}$$

Although, we have the identification $\mathbb{E}_m \wedge \mathbb{E}_n \cong \mathbb{E}_{m+n}$ it is most natural to index the wedge-summands of this space, which are $m+n$-spheres by pairs of integers.

**Example 2.17** (Reduced cones of shrinking wedges)**.** Since $C_*X = X \wedge [0,1]$ is the reduced cone over $(X, x_0)$, we have

$$C_*\left(\widetilde{\bigvee}_{j\in J}X_j\right) \cong \left(\widetilde{\bigvee}_{j\in J}X_j\right) \wedge [0,1] \cong \widetilde{\bigvee}_{j\in J}(X_j \wedge [0,1]) \cong \widetilde{\bigvee}_{j\in J}C_*X_j$$

In other words, the reduced cone over a shrinking wedge is canonically homeomorphic to the shrinking wedge of the reduced cones of the wedge-summand spaces. Hence, a based map $f : \widetilde{\bigvee}_{j\in J}X_j \to Y$ is null-homotopic rel. basepoint if and only if it extends to a map $g : \widetilde{\bigvee}_{j\in J}C_*X_j \to Y$.

## 3. An infinite analogue of the smash product

Let $\{(X_i, x_i)\}_{i\in\mathbb{N}}$ be a sequence of based spaces. According to Remark 2.3, we may identify the shrinking wedge $\widetilde{\bigvee}_{i\in\mathbb{N}}X_i$ canonically as a subspace of the infinite direct product $\prod_{i\in\mathbb{N}} X_i$ where $X_i$ is identified with

$$\{x_1\} \times \{x_2\} \times \cdots \times \{x_{i-1}\} \times X_i \times \{x_{i+1}\} \times \cdots$$

and the basepoint is $b_0 = (x_1, x_2, x_3, \dots)$. In this way, we may consider the quotient space $\prod_{i\in\mathbb{N}} X_i\big/\widetilde{\bigvee}_{i\in\mathbb{N}}X_i$. Let $q : \prod_{i\in\mathbb{N}} X_i \to \prod_{i\in\mathbb{N}} X_i\big/\widetilde{\bigvee}_{i\in\mathbb{N}}X_i$ be the canonical quotient map. For each $m \in \mathbb{N}$, we also have a quotient map $q_m : \prod_{i=1}^{m} X_i \to \prod_{i=1}^{m} X_i\big/\bigvee_{i=1}^{m} X_i$. Notice that we have a commutative diagram where the top row consists of projection maps and the bottom row consists of maps $g_{m+1,m}$ induced by $q_m \circ \pi_{m+1,m}$ on the quotient space. We will let $Y_m = \prod_{i=1}^{m} X_i\big/\bigvee_{i=1}^{m} X_i$ with the point $w_m \in Z_m$ representing the image of $\bigvee_{i=1}^{m} X_i$. We also denote $Y = \varprojlim_{n} (Y_m, g_{m+1,m})$.

$$\begin{tikzcd}
\prod_{i\in\mathbb{N}} X_i \arrow[d, "q"] \arrow[rr, bend left, "p_3"] \arrow[rrr, bend left, "p_2"] \arrow[rrrr, bend left, "p_1"] & \cdots \arrow[r, "\pi_{4,3}"'] & X_1 \times X_2 \times X_3 \arrow[r, "\pi_{3,2}"'] \arrow[dd, "q_3"] & X_1 \times X_2 \arrow[r, "\pi_{2,1}"'] \arrow[dd, "q_2"] & X_1 \arrow[dd, "q_1"] \\
\prod_{i\in\mathbb{N}} X_i\big/\widetilde{\bigvee}_{i\in\mathbb{N}}X_i \arrow[d, "\phi"] & & & & \\
Y \arrow[rr, bend right, "g_3"] \arrow[rrr, bend right, "g_2"] \arrow[rrrr, bend right, "g_1"] & \cdots \arrow[r, "g_{4,3}"] & \frac{X_1 \times X_2 \times X_3}{X_1 \vee X_2 \vee X_3} \arrow[r, "g_{3,2}"] & \frac{X_1 \times X_2}{X_1 \vee X_2} \arrow[r, "g_{2,1}"] & \frac{X_1}{X_1} = *
\end{tikzcd}$$

Certainly, $q$ is constant on the fibers of $q_m \circ p_m$ and so there exists a unique map $\phi_m : \prod_{i\in\mathbb{N}} X_i\big/\widetilde{\bigvee}_{i\in\mathbb{N}}X_i \to Y_m$ such that $\phi_m \circ q = q_m \circ p_m$. Notice that

$$g_{m+1,m} \circ \phi_{m+1} \circ q = g_{m+1,m} \circ q_{m+1} \circ p_{m+1} = q_m \circ \pi_{m+1,m} \circ p_{m+1} = q_m \circ p_m = \phi_m \circ q.$$

Since $q$ is surjective, this gives $g_{m+1,m} \circ \phi_{m+1} = \phi_m$ for all $m \in \mathbb{N}$. Hence, there is a unique, continuous map $\phi$ such that if $g_m : Y \to Y_m$ are the projections of the limit, then $g_m \circ \phi = \phi_m$.

**Theorem 3.1.** *If $X_m$ is compact Hausdorff for all $m \in \mathbb{N}$, then*

$$\phi : \prod_{m\in\mathbb{N}} X_m \Big/ \widetilde{\bigvee}_{m\in\mathbb{N}} X_m \to \varprojlim_m \left( \prod_{i=1}^m X_i \Big/ \bigvee_{i=1}^m X_i, g_{m+1,m} \right)$$

*given by $\phi(a) = (\phi_1(m), \phi_2(m), \phi_3(m), \dots)$ is a homeomorphism.*

*Proof.* First, we check that $\phi$ is bijective. Let $y = (z_1, z_2, z_3, \dots) \in Y$. If $y = (w_1, w_2, w_3, \dots)$ then clearly $\phi(q(b_0)) = y$. If $z_m = w_m$ for some $m$, then $z_i = w_i$ for all $i \leqslant m$. Therefore, if $y \neq (w_1, w_2, w_3, \dots)$, then there exists $M \in \mathbb{N}$ such that $z_m \neq w_m$ for all $m \geqslant M$. In particular, for all $m \geqslant M$, there is a unique point $t_m \in Y_m$ such that $q_m(t_m) = z_m$. Since $g_{m+1,m}(z_{m+1}) = z_m$ and $g_{m+1,m}$ agrees with $\pi_{m+1,m}$ on $Y_{m+1}$, it follows that $\pi_{m+1,m}(t_{m+1}) = t_m$ for all $m \geqslant M$. Recursively define $t_i = \pi_{i+1,i}(t_{i+1})$ for $1 \leqslant i \leqslant M - 1$. Now $(t_1, t_2, t_3, \dots) \in \varprojlim_m (\prod_{i=1}^m X_i, \pi_{m+1,m}) = \prod_{m\in\mathbb{N}} X_m$ and $\phi(q(t_1, t_2, t_3, \dots)) = y$. This shows $\phi$ is onto.

Suppose $a = (a_1, a_2, a_3, \dots)$ and $b = (b_1, b_2, b_3, \dots)$ are distinct elements of $\prod_{m\in\mathbb{N}} X_m$. Suppose $a, b \notin \widetilde{\bigvee}_{m\in\mathbb{N}} X_m$. Then

- there exists $m_0 \in \mathbb{N}$ such that $a_{m_0} \neq b_{m_0}$,
- there exists $j_1, j_2$ such that $a_{j_1} \neq x_{j_1}$ and $a_{j_2} \neq x_{j_2}$,
- there exists $k_1, k_2$ such that $b_{k_1} \neq x_{k_1}$ and $b_{k_2} \neq x_{k_2}$.

Let $M = \max\{m_0, j_1, j_2, k_1, k_2\}$. Then $\phi_M(a) \neq \phi_M(b)$, proving that $\phi(a) \neq \phi(b)$. The other case to consider is when $a \notin \widetilde{\bigvee}_{m\in\mathbb{N}} X_m$ and $b \in \widetilde{\bigvee}_{m\in\mathbb{N}} X_m$. Then

- there exists $j_1, j_2$ such that $a_{j_1} \neq x_{j_1}$ and $a_{j_2} \neq x_{j_2}$,
- $\phi_m(b) = w_m$ for all $m \in \mathbb{N}$.

Let $M = \max\{j_1, j_2\}$. Then $\phi_M(a) \neq w_m = \phi_M(b)$, proving $\phi(a) \neq \phi(b)$.

As a quotient of a compact space, $\prod_{i\in\mathbb{N}} X_i \Big/ \widetilde{\bigvee}_{i\in\mathbb{N}} X_i$ is compact. Moreover, since $\prod_{i=1}^m X_i$ is compact Hausdorff, and $\bigvee_{i=1}^m X_i$ is closed in the $T_4$ space $\prod_{i=1}^m X_i$, it follows that each $Y_m$ is Hausdorff. Thus the inverse limit $Y$ is Hausdorff. Since $\phi$ is a continuous bijection from a compact space to a Hausdorff space, it is a homeomorphism. □

**Remark 3.2.** Each bonding map $g_{m+1,m} : Y_{m+1} \to Y_m$ admits a canonical section $s_{m,m+1} : Y_m \to Y_{m+1}$. Therefore, we may identify a sequence of closed subspaces of $Y$:

$$* \subseteq \frac{X_1 \times X_2}{X_1 \vee X_2} \subseteq \frac{X_1 \times X_2 \times X_3}{X_1 \vee X_2 \vee X_3} \subseteq \frac{X_1 \times X_2 \times X_3 \times X_4}{X_1 \vee X_2 \vee X_3 \vee X_4} \subseteq \cdots$$

The union $Y_{fs} = \displaystyle\bigcup_{m\in\mathbb{N}} Y_m = \bigcup_{m\in\mathbb{N}} \prod_{i=1}^m X_i \Big/ \widetilde{\bigvee}_{i=1}^m X_i$ is dense in $Y$.

## References


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